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ON A CLASS OF BILATERAL GENERATING FUNCTIONS FOR HERMITE POLYNOMIALS OF TWO VARIABLES

G. K. PANJA* AND D. K. BASU**

(Received 20.11.2000 and in revised form 10.11.2001)

ABSTRACT

A new class of bilateral generating functions for Hermite polynomials of two variables $H_n(x, y)$ is obtained. Applications of our result are pointed out.

Key word & phrases: Special functions & bilateral generating function.
Mathematics Subject Classification (2000): 33C45

INTRODUCTION

Noticing the existence of the following type of generating function of Hermite polynomials of two variables $H_n(x, y)$ [5]

$$e^{2xt-yt^2}\left(1-\frac{yt}{x}\right)^{2\alpha} H_\alpha\left[x\left(1-\frac{yt}{x}\right)^{-1}, y\left(1-\frac{yt}{x}\right)^{-4}\right] = \sum_{n=0}^{\infty} \frac{H_{\alpha+n}(x, y)t^n}{n!} \tag{1}$$

where α is a nonnegative integer, we are led to investigate a more general class of generating function by Lie group - theoretic method.

The main result of our investigation is the following theorem ;

Theorem : If there exists a liner generating function of the form

$$G(x, y, t) = \sum_{n=0}^{\infty} \partial_n \mathrm{H}\alpha + \mathrm{n}(x, y)\mathrm{t}^n \tag{2}$$

then the following new bilateral generating function [3] will exist.

$$\left(1-\frac{yz}{x}\right)^{2\alpha} \exp(2xz-yz^2)\mathrm{G}\left[x\left(-\frac{yz}{x}\right)^{1}, y\left(1-\frac{yz}{x}\right)^{4}, tz\left(1-\frac{yz}{x}\right)^{2}\right]$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{a_n}{(k-n)!} H_{\alpha+\mathrm{k}}(x, y)t^n z^k \text{ with } \left|\frac{yz}{x}\right| < 1 \tag{3}$$

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GROUP THEORETIC DISCUSSION

For $H_n(x, y)$ we have the following differential operator [5]

$$B = z\left(y\frac{\partial}{\partial x} + \frac{4y^2}{x}\frac{\partial}{\partial y} - \frac{2yz}{x}\frac{\partial}{\partial z} + 2x \right) \tag{4}$$

such that [3]

$$B\left[H_n(x.y)z^n\right] = H_{n+1}(x,y)z^{n+1} \tag{5}$$

The extended form of the transformation group generated by B is [3]

$$(\exp bB)f(x,y,z) = \exp\left\{2bxz\left(1-\frac{byz}{2x}\right)\right\} f\left[x\left(1-\frac{byz}{x}\right)^{-1}, y\left(1-\frac{byz}{x}\right)^{-4}, z\left(1-\frac{byz}{x}\right)^{2}\right] \tag{6}$$

DERIVATION OF THE GENERATING FUNCTION

Let us consider the generating function

$$G(x, y, t) = \sum_{n=0}^{\infty} a_n H_{\alpha+n}(x,y)t^n \tag{7}$$

Multiplying both sides by t^α, we get

$$t^\alpha G(x, y, t) = \sum_{n=0}^{\infty} a_n H_{\alpha+n}(x,y)t^{\alpha+n} \tag{8}$$

Replacing t by tz, we get

$$t^\alpha z^\alpha G(x,y,tz) = \sum_{n=0}^{\infty} a_n H_{\alpha+n}(x,y)z^{\alpha+n}t^{\alpha+n} \tag{9}$$

Operating both members of eqn. (9) by exp (bB)

$$\{\exp(bB)\}[t^\alpha z^\alpha G(x,y,tz)] = \{\exp(bB)\}\sum_{n=0}^{\infty} a_n H_{\alpha+n}(x,y)z^{\alpha+n}t^{\alpha+n}$$

The left hand side of the last equation becomes

$$t^{\alpha} z^{\alpha} (1 - \frac{byz}{x})^{2\alpha} \exp \left\{2bxz(1 - \frac{byz}{2x})\right\} G\left[x(1 - \frac{byz}{x})^{-1}, y(1 - \frac{byz}{x})^{-4}, tz(1 - \frac{byz}{x})^{2}\right]$$

On the other hand using (5) the right hand side is reduced to

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_n \frac{b^k}{k!} \mathrm{H}_{\alpha+n+k}(x, y) z^{\alpha+n+k} t^{\alpha+n}$$

Hence we get

$$\left(1 - \frac{byz}{x}\right)^{2d} \exp\left\{2bxz(1 - \frac{byz}{2x})\right\} G\left[x\left(1 - \frac{byz}{x}\right)^{-1}, y\left(1 - \frac{byz}{x}\right)^{-4}, tz\left(1 - \frac{byz}{x}\right)^{2}\right]$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_n \frac{b^k}{k!} H_{\alpha+n+k}(x, y) z^{n+k} t^{n} \tag{10}$$

Now putting $b = 1$ we get eqn. (3) using Lemma 10 of Rainville [6, pp. 56, 57]

APPLICATIONS

If we consider $a_n = \frac{1}{n!}$ then $G(x, y, t) = \sum_{n=0}^{\infty} \frac{H_{\alpha+n}(x, y) t^n}{n!}$

Also from [5] we have

$$G(x, y, t) = e^{2xt - yt2} \left(1 - \frac{yt}{x}\right)^{2\alpha} H_{\alpha}\left[x\left(1 - \frac{yt}{x}\right)^{-1}, y\left(1 - \frac{yt^{-4}}{x}\right)\right] \tag{11}$$

Again, we obtain from [5]

$$G\left[x(1 - \frac{yz}{x})^{-1}, y(1 - \frac{yz}{x})^{-4}, tz(1 - \frac{yz}{x})^{2}\right] = e^{2xzt - 2ytz^2 - yt^2z^2} \left(\frac{x - yz - ytz}{x - yz}\right)^{2\alpha} H_{\alpha}\left[x\left\{1 - \frac{yz(1+t)}{x}\right\}^{-1}, y\left\{1 - \frac{yz(1+t)}{x}\right\}^{-4}\right] \tag{12}$$

From equations (3) and (12) we obtain

$$\exp\left[2xz(1+t) - yz^2(1+t)^2\right]\left\{1 - \frac{yz}{x}(1+t)\right\}^{2\alpha} \mathrm{H}_{\alpha}\left[x\left\{1 - \frac{yz}{x}(1+t)\right\}^{-1} y\left\{1 - \frac{yz}{x}(1+t)\right\}^{-4}\right] = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{1}{n!(k-n)!} H_{\alpha+k}(x, y) t^n z^k \tag{13}$$

Now applying the relation $H_n(x,y) = y^{n/2}\ H_n(x/\sqrt{y})$ we get on putting y = 1

$$\exp\ [2xz(1+t) - z^2(1+t)^2]H_\alpha[x - z(1+t)] = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{1}{n!(k-n)!}H_{\alpha+k}(x)t^n z^k \qquad (14)$$

which is a new bilateral generating function for Hermite polynomial of a single variable.

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BHĀSKARA'S METHODS FOR COMPUTING MEAN PLANETARY POSITIONS[†]

S. L. SINGH* AND RAMESH CHAND**

(Received 15.01.2001)

ABSTRACT

In this paper we discuss several methods from *Siddhānta Śiromaṇi* (SiS) for computing mean planetary position of planents. A few applications are also discussed.

INTRODUCTION

In his *Grahaganita* (c. 1150 A.D.), *Ācārya Bhāskara* (generally called *Bhāskara* II) (b. 1114 A.D.) (henceforth called *Ācārya* in this paper) has given sevreal computational methods for the determination of planetary position at a particular place and time. The classical presentation of *Ācārya's* highly respected formulae are apparently not very friendly to a modern astronomer. In this paper, we attempt a systematic presentation of these methods which can be used very conveniently even by a sophomore in astronomy. Indeed, here we present six methods concerning the mean position and illustrate them by examples.

[†] A part of the talk given at the National Symposium on Ancient Science in India, organized during the third annual conference of the *Vijnāna* Parishad of India [dedicated to the 800th death anniversary of *Bhāskarācārya* (1114-1193 A.D.)], H.N.B Garhwal University, Srinagar (Garhwal), May 25-26, 1993.

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COMPUTATION OF MEAN POSITION

1. First Method

द्युचरचक्रहतो दिनसञ्चयः कूहहृतो भगणादि फलं ग्रहः।
दशशिरःपुरि मधयमभास्करे क्षितिजसंनिधिगे सति मध्यमः।।

[SiS,] *Grahaṇāyanādhyāya*]

This means (see [2, p. 31])

The *ahargaṇa* multiplied by the number of sidereal revolutions of the planet and divided by the number of civil days in a *kalpa* gives the mean position of the planet at the time when the mean sun is very nearly at the eastern horizon in *Shrilanka.*

This, using notations, may be presented as follows :

Mean longitude of a planet in *Shrilanka* = (R x A) ÷ C, wherein R is the sidereal revolutions of a planet in a *Kalpa,* A = *ahargaṇa* and C = civil days in the present *Kalpa.* For details of *ahargaṇa, Kalpa* and other technical terms one may refer to [1-11] and [13].

Example. Compute the mean longitude of the planets on May 25, 1993.

Following standard techniques of calculating *ahargaṇa* (cf. Lahiri [7] and also [13]). A (*ahargaṇa*) on May 25, 1993, A=5168 x 360 + 3 x 30 + 30 + 25 = 1860667 civil days. Therefore, on May 25, 1993, mean longilude of the Sun.

= (432 x 10^7 x 1860667) / 1577916450000 revolutions

= 5094. 110933 revolutions

= 5094 revolutions / 1 *Rās'i* / 9^0/ 56'/ 9.68

mean longitude of the Moon

= (577533 x 10^5 x 1860667) ÷ 1577916450000 revolutions

68102.24896 revolutions

= 68102 revolutions / 2 *Ṙāsi* / 29^0/ 37'/ 32.16 ";

mean longitude of Mars

$$= \frac{229828522 \times 1860667}{1577916450000} + \textit{Dhruvakās}$$

= 271. 0120338 Revolutions +11 *Rāsi* / 29^0/3'/50"

= 272 Revolutions / o *Rāsi* / 3^0/ 23'/ 45.4048".

Similar computations may be made for other planets (see, for example, [13]).

2. Second Method

कोट्याहतैरङ्ककृतेन्दुविश्वैः 13 1490000000
न्यूनाहशेषे विहृते लवाद्यम्।
रविघ्नतिथ्याढ्यमनेन युक्तो
रविर्विधुः स्याद्विधुरूनितोऽर्क।।

[SiS, *Grahaṇāyanādhyāya*]

The *avama-śeṣa* divided by 13149×10^7 in (degrees) is to be added to twelve times the elapsed *tithis* and the result added to the sun's position gives the position of the moon. Conversely the position of the sun can be obtained from the position of the moon (see,[2, p. 32])

This means Moon's longitude = $(x+12\,Y)^0+R^0/13149 \times 10^7$
(wherin x = sun's longitude, Y = elapsed *tithis* up to the concerned day,
R = *avama s'easa*).

For the computation of elapsed *tithis* till May 25, 1993, we as fallows:
D (elapsed solar days till May 25, 1993)

$= \{[3179 + (1993 - 78)] \times 12 + 2\} \times 30 + 3.$

$= 1833903.$

Elapsed intercalary months

= DA/S, (where A = *adhikamāsa* of the *Kalpa*, S= number of solar days in *Kalpa*)

$= (1833903 \times 15933 \times 10^5) \div 15552 \times 10^8$

$= 1878.83791$

= 1878 (Discarding the remainder as per rule).

Y (elapsed *tithis*)

= D+elapsed intercalary months x 30

$= 1833903 + 1878 \times 30$

$= 1890243$

Since $R = (Y \times ot) \div T$ (where Ot = Omitted *tithis* in *kalpa* and T *tithis* in *Kalpa*) we have :

$R = (1890243 \times 2508255 \times 10^4) \div 1602999 \times 10^6$

$= 29577.13294$

The mean longituded of the Moon on May 25, 1993

= $(x + 12\,Y)^0 + R^0 / 13149 \times 10^7$

= (5094. 110933 + 12 x 1890243 / 360) revolutions + $(29577.\ 13294/\ 1349 \times 10^7)^0$

= 68102 Revolutions/ 2 Rāśi $15^0/56'/5.2808067''$.

3. Third Method :

यथा यथाऽधिमासकाऽवमेन्दुमासपूर्वकाः
परस्परं युतोनिता भवन्ति खेटपर्य्ययाः।।
त एवं सूर्य्यसावनद्युपिराडतोऽनुपातजाः
तथा तथा युतोनिता भवन्ति तेऽथवा ग्राहाः।।

[Sis *Grahānayanādhyāya*]

Even as the sums or differences of two or more of the numbers of *adhimasas*, *avama* (*ksayāhās*), lunations etc. give the number of sidereal revolutions of the planets, the sums or differences of two or more of the positions of the imaginary planets which go by the names *adhimāsa – graha*, *ksayahagraha* etc. computed out of the numbers of these *adhimāsas ksayahas* etc. give the respective planetary positions. (See [2, p. 39]).

According to this, longitude of the moon on May 25, 1993

= 13 x Revolutions of the sun + Intercalary- months

= 13 x 5094. 11093 + 1878. 830791

= 68102.272911 Revolutions

= 68102 Revolutions / 3 Rās'i / $8^0/14'/52.656''$.

4. Fourth method :

अहर्गणात् कुक्षिनवाङ्कः 9921 निन्घात्-
नवेन्दुवेदेषुहुताश 35419 लब्ध्या।
अहर्गणो गोऽक्षघृतीन्दु 11859 निघ्नो
विवर्जितः स्युर्गतयोजनानि।
स्वया स्वया तानि पृथक् च कक्षया
हृतानि वा स्युर्भगणादिकाः ग्रहाः।

[Sis, *Madhyadhikara Madhyadhikara*]

The *ahargaṇa* multiplied by 11859 decreased by the quotient obtained by dividing the product of the *ahargaṇa* and 9921 by 35419 given the distance covered by a planet in *yojanas*. These *yojanas* divided by the circumference of the planet's orbit gives the fraction of a revolution and the integral number of revolutions made. (See [2. p. 50-51]).

This means distance covered by a planet in *Yojnas*

= A x 11859 - A x 9921 / 35419,

where A is the required *ahargaṇa* on the concerned day on May 25, 1993 and A = 1860667 days.

Thus the distance covered by a planet

= 1860667 (11859 - 9921/ 35419) Yojanas

= 1860667 (11858.7199) Yojanas

The mean longitude of the sun on May 25, 1993

= (Distance covered by the planet in the required *ahargaṇa*)

÷ (circumference of the planet's orbit)

= (1860667 x 11858.7199) ÷ 43314975 Revolutions

(wherein 4331497.5 Yojanas is the circumference of the Sun's orbit)

= 5094.110924 Revolutions

=5094.110924 Revolutions/ 1 *Rās'i* / $9^0/55'/57.504''$

The mean longitude of the moon on May 25, 1993

= (1860667 x 11858.7199)/324000 Revolutions

(Wherin 324000 Yojanas is the circumference of the moon's orbit)

= 68102.24931 Revolutions

= 68102 Revolutions / 2 Rās'i / $29^0/45'/5.76''$

5. Fifth method

(V.1) Longitude of the Sun

दिनगणो निजषष्टिलवान्वितो
भवति तिगमरुचिः स लावादिकः।
गुणगुणाद् द्युगणादथ भाजिताद्
यमयमेः 22 कलिकादिफलान्वितः।।

[Sis, *Pratyabada s'uddhi* in *madhyādhikāra*]

The number of the days in the *ahargaṇa* is to be diminished by 1 / 60th part of itself to obtain the number of degrees and fractions thereof; then the ahargana multiplied by three and divided by 22 gives the minutes and fraction there of. Adding the two results, we get the reqired position of the Sun. (See[2.p.64])

Accordingly the mean longitude of the Sun
= (*Ahargaṇa* - *Ahargaṇa*/60^0) + (3 x *Ahargaṇa*/22)
= *Ahargaṇa* [$(59/60)^0$ + (3/22)']
The mean longitude of the sun of May 25, 1993
= 1860667 [$(59/60)^0$ + (3/22)']
=5094.124033 Revolution
=5094 Revolution/ 1 Rās'i / /14^0/39'/6.768"

(V.2) Longitude of the Mars :

दिनगणार्द्धमघो गुणसंगुणम्
द्युगणसप्तदशांशविवर्जितम्।
लवकलादिफलद्वयसंयुत:
क्षितिसुतध्रुवकः क्षितोजो भवेत्।।

[Sis, *Pratyabada s'uddhi* in
Madhyādhikāra]

" Half of the number of days gives the degrees; half the number of days multiplied by 3 gives the number of minutes "; from this subtracting 17th part of *Ahargaṇa* gives the minutes and adding the *Drhuvkā* at the beginning of the solar year, gives the mean position of Mars (See [p.65]).

Thus the longitude of the Mars
= (*Aharagaṇa* / $2)^0$ + (3 *Aharagaṇa* / 2)'- (*Aharagaṇa* / 17)' + *Dharuvaka.*
The longitude of Mars of May 25, 1993

= (1860667/2 + 3 x 1860667/2 x 60-1860667/17 x 60) + *Dhruvaka*
= 2708 Revolutions / 145^0/59'/23.8992"+11 Rās'i /29^0/3'/50"
=2709 Revolutions/4 Rās'i / 24^0/62'/73.8992".

(V.3) Longitude of Mercury's ***S'ighrocca*** **:**

दिनगणः कृतसंगुणितः पृथग्।
गुणगुणः खगुणेन्दुभिरुद्धृतः।
फलयुतः खलु तेन लवादिना
बुधचलं भवति ध्रुवकोऽन्वितः।।

[Sis, *Pratyabada s'uddhi* in *Madhyādhikāra*]

The *Ahargaṇa* multiplied by 4 and 4/130, gives respectively the positive degrees and minutes to be added to the *Dhruvakās* of Mercury's *s'ighrocca*.

Thus the longitude of Mercury's *s'ighrocca*
=(4 *Ahargaṇa*)^0 + (4 *Ahargaṇa* 3/130)' + *Dhruvaka*
The mean longitude of Mercury's s'ighrocca on May 25, 1993

= (4 x 1860667 + 4 x 1860667 x 3/ 130)^0 + 11 *Rās'i* /27^0-/24'/29"
= 21151. 172 Revolutions + 11 *Rās'i* / 27^0/24'/29"
= 21152 Revolutions / 58^0/2'/0"

(V.4) Logitude of Jupiter :

द्युमणिभिः कनुगैर्द्युगणो हृतो
लवकलाः स्वमृणं ध्रुवके गुरुः।

[Sis, *Pratyabada s'uddhi* in *Madhyādhikāra*]

The *ahargaṇa* divided by 12 and by 71 gives respectively the positive degrees and negative minutes to be added to the *Dhruvaka* of Jupiter. (See[2, p. 66]).

Thus the longitude of Jupiter
= (*Ahargaṇa*/12)^0 - (*Ahargaṇa*/71)' + *Dhruvakās*
The mean longitude of jupiter on May 25, 1993
= (1860667/12)^0 - (1860667/71)' + 11 *Rās'i* / 29^0/27'/36"
= 430 Revolutions / 5 *Rās'i* /28^0/16'/1.30244"

(V.5) Longitude of Venus ***S'ighrocca*** **:**

ऋतुभिरक्षदिनैर्दशसङ्गुणात्
फललवाः स्वमृणं ध्रुवके सितः।।

[SiS. *Pratyabada S'uddhi* in *Madhyādhikāra*]

The *Ahargaṇa* multiplied by 10 and divided by 6 and 155 respectively gives degrees positive and negative to be added to the *Dhruvaka* of Venus to give his position. (See, [2, p. 66])

Thus the mean longitude of Venus *S'ighrocca*
= (10 *Ahargaṇa*/ 6)^{0} - (10 *Ahargaṇa*/ 155)^{0} + *Dhruvaka*

- The mean longitude of Venus *S'ighrocca* on May 25, 1993
= 1860667 [10/6 - 10/155]^{0} + 11 *Rās'i*/29^{0}/42'/14"
= 8280. 7462 Revolutions + 11 *Rās'i*/29^{0}/42'/14"
= 8281 Revolutions / 8 *Rās'i*/27^{0}/20'/17.7536"

(V.6) Longitude of Saturn

द्विनोघ दिनोघः पृथगक्षभक्तो
लिप्ता विलिप्ता ध्रुवके स्वमाकिंः।

[SiS, *Pratyabada S'uddhi* in *Madhyādhikāra*]

This means that the longitude of Saturn
= (2 x *Ahargaṇa*)' + (2 x *Ahargaṇa*/5)" + *Dhruvaka*
The mean longitude of Saturn on May 25, 1993
= (2 x 1860667)' + (2 × 1860667 / 5)" +11 Rās'i /28^{0}/46'/34"
= 172 Revolutions/308^{0}/58'/26.78736"+11 Rās'i + 28^{0}/46'/34"
= 173 Revolutions/10 Rās'i /7^{0}/45'/0.78736"

(V.7) Longitude of Moon's Apogee :

दिग्भिर्गजेभैश्च हृतो दिनौघः।
क्षेप्यो ध्रुवांशेषु भवेद्विधूच्चम्।।

[SiS, *Pratyabada S'uddhi* in *Madhyādhikāra*]

The *Ahargaṇa* divided successively by 10 and 88 and added gives the degrees to be added to the *Dhruvaka* of the apogee of the moon to obtain its position. (See [2, p.67]

This means that the longitude of Moon's apogee

= (*Ahargaṇa* / 10)^{0} + (*Ahargaṇa*/ 88)^{0} + *Dhruvaka*

This means longitude of Moon's apogee on May 25, 1993

$= 1860667\ (1/10 + 1/88)^0 + \textit{Dhruvaka}$

$= 575.58512$ Revolutions $+ 4$ *Rās'i* $/50^0/29'/46''$

$= 575$ Revolutions $/ 11$ *Rās'i* $/6^0/8'/21.52''$

(V.8) Longitude of Lunar Node:

ताड़ित खदहनैर्दिनसङ्घः षट्कषट्कशरहृत् फलमंशाः।
स्वं ध्रुवे कुमुदिनीपतिपातो राहुमाहुरिह केऽपि तमेव।।

[Sis. *Pratyabada S'uddhi* in *Madhyādhikāra*]

The *Ahargaṇa* multiplied by 30 and divided by 566 gives the number of degrees to be added to *Dhruvakā* to give the position of *Rahu* (lunar node). (See, [2, P.68]). This means that the longitude of lunar node

$= (30\ \textit{Ahargaṇa}/566)^0 + \text{Dhruvakā}$

The longitude of the lunar node on May 25, 1993

$= (30 \times 1860667/566)^0 + 5$ *Rās'i* $/ 3^0/ 12'/58''$

$= 273.94979$ Revolutions $+ 5$ Rās'i $/ 3^0\ 12'/ 58''$

$= 273$ Revolutions$/ 341^0 / 55'/ 27.84'' + 5$ Rās'i $/ 3^0/ 12'/ 58''$

$= 274$ Revolutions $/ 4$ Rās'i $/ 15^0/ 8'/ 25.84''$

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AN IDEALISED MODEL OF AN M.H.D. SPHERICAL SHOCK WAVE IN SELF-GRAVITATING GAS

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ABSTRACT

An analytical solution of a shock, in a spherically symmetric model incorporation a self-consistent azimuthal magnetic field has been studied using chester-chisnell-whitham (C.C.W.) method. The shock is assumed to advance into a self-gravitating and conducting gas of spatially decreasing density and pervaded by an idealised spatially decreasing magnetic field.

Numerical solutions are obtained for shock velocity and shock strength.

KEY WORDS: Spherically symmetric model, Chester-Chisnell- Whitham method, shock waves.

INTRODUCTION

Summers [10] has constructed an idealised model of a magnetohydrodynamic spherical blast wave applied to a flare produced shock in the solar wind. He has omitted gravity in his problem and solved it using the similarity method. Recently Singh and Mishra [7, 5], Singh and Pandey [8, 9], Yadav [12], Kumar and Prakash [3] and Kumar and Saxena [4], have discussed different types of problems of cylindrical shock waves allowing the effects of gravitational force, magnetic field and also in rotating atmosphere. They all used Chester [1]- Chisnell [2]-Whitham [11] (C.C.W.) method for their solution. None has attempted these problems of shock wave in spherical symmetry in magnetogasdynamics using the Whitham's rule. We have investigated the analytical solutions for the propagation of spherical shock wave in selfgravitating gas of spatially decreasing density in the presence of spatially decreasing magnetic field. The problem has been studied for weak shock with weak and strong magnetic field as well as strong shock with strong magnetic field.

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Analytical expressions have been obtained for shock velocity and shock strength with other variables also.

According to Summers [10] and Rosenau and Frankenthal [6] we have taken the density distribution ρ_0^{*} ahead of the shock to vary as an inverse power or radial distance from the point of explosion, i.e.

$$\rho_o = \rho_o^{\,r-w} \tag{1}$$

where ρ_o and w are constants.

The magnetic field H_0 is replaced by a physically unrealisable idealised field such that lines of force lie on hemisphere whose centre is the explosion point as shown in the figure one given in the paper of Summers [10]. The magnetic field distribution is assumed to be of similar form, i.e.

$$H_0 = H_c r^{-\delta} \tag{2}$$

and directed tangential to the advancing shock front. Equation (2) is introduced purely to give the mathematics of the problem tractable and its in correct form will not substantially affect the overall validity of the present model.

BASIC EQATIONS AND BOUNDARY CONDITIONS

The equations governing the spherically symmetrical flow of a gas under the influence of its own gravitation in the presence of azimuthal magnetic field following Summer's, are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{GM}{r^2} + \frac{\mu H}{\rho}\frac{\partial H}{\partial r} + \frac{\mu H^2}{\rho r} = 0 \tag{3}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0 \tag{4}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} - a^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) = 0 \tag{5}$$

$$\frac{\partial m}{\partial r} - 4\pi r^2 \rho = 0 \tag{6}$$

$$\frac{\partial H}{\partial t}+u\frac{\partial H}{\partial r}+H\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)=0 \tag{7}$$

where $m\,(r,t)$, $u\,(r,t)$, $\rho\;(r,t)$ and $H\,(r,t)$ denote, respectively, the mass inside a sphere of radius r, the velocity, the pressure, the density and transverse magnetic field at a distance r at time t. μ is magnetic permeability and

$$a^2=\frac{\gamma p}{\rho}$$

The magnetohydrodynamic conditions can be written in terms of a single parameter $N=\frac{\rho_1}{\rho_0}$ as

$$\rho_1=N\rho_0, H_1=NH_0, u_1=(1-1/N)U$$

$$U^2=\frac{2N}{(\gamma+1)-(\gamma-1)N}[a_0^2+b_0^2\{2-\gamma)N+\gamma\}] \tag{8}$$

$$p_1=p_0+\frac{2\rho_0(N-1)}{(\gamma+1)-(\gamma-1)N}\left[a_0^2+\frac{\gamma-1}{4}b_0^2(N-1)^2\right]$$

Where 0, and 1, respectively stand for the states just ahead and just behind the shock front. U is shock velocity, a_0 is the sound speed $(\gamma p_0/\rho_0)^{1/2}$, and b_0 is the Alfven speed $(\mu H_0^2/\rho)^{1/2}$.

1. Weak Shocks

For very weak shock we take the parameter as

$$\frac{\rho_1}{\rho_0}=N=1+\varepsilon \tag{9}$$

Where $\varepsilon \ll 1$. Now consider the two cases of weak and strong magnetic fields.

Case -I

For weak magnetic field $b_0^2/a_0^2 \ll 1$ under this condition and (9), the boundary conditions (8) for very weak shocks reduces to

$$\rho_1=\rho_0(1+\varepsilon),\ H_1=H_0(1+\varepsilon), u_1=\varepsilon a_0 \tag{10}$$

$$U = (1+\frac{\gamma+1}{4}\varepsilon)a_0, p_1 = p_0(1+\gamma\varepsilon)$$

Case - II

For strong magnetic field $b_0^2 / a_0^2 >> 1$, using this conditions and equation (9), the boundary condition (8) reduces to

$$\rho_1 = \rho_0(1+\varepsilon),\ H_1 = H_0(1+\varepsilon), u_1 = \varepsilon b_0$$

$$U = \left(1+\frac{3}{4}\epsilon\right)b_0,\ p_1 = p_0(1+\gamma\varepsilon) \tag{11}$$

2. Strong Shocks

In the limiting case of a strong shocks p_1 / p_0 is large and in the presence of the magnetic case this can be brought about in two ways.

Case-I

The purely non-magnetic way when $N \rightarrow \frac{\gamma+1}{\gamma-1}$

Case-II

When $b_0^2 >> a_0^2$ or when $\mu H_0^2 >> p_0$, that is when the ambient magnetic pressure is large compared with the ambient fluid pressure. In terms of N the boundary conditions (8) now become

$$\rho_1 = N\rho_0, N_1 = NH_0, u = (1-1/N)U,$$

$$\frac{p_1}{p_0} = X(N)\frac{U^2}{a_0^2} + Y(N) \tag{12}$$

Where

$$X(N) = \frac{\gamma(\gamma-1)(N-1)^3}{2N[2-\gamma]N+\gamma]}$$

$$Y(N) = \frac{N(\gamma+1)-(\gamma-1)}{(\gamma+1)-(\gamma-1)N}$$

For diverging shocks the characteristic form of system of equations (3) to (7)- that is, the form in which equation contains derivatives in only one direction in (r, t) plane is

$$dp + \mu H dH + \rho c du \frac{2\rho c^2 u}{u+c}\frac{dr}{r} - \frac{\mu H^2 (u-c)}{u+c}\frac{dr}{r} + \frac{\rho c G m}{u+c}\frac{dr}{r^2} = 0 \tag{13}$$

Where $c^2 = a^2 + b^2 = \frac{\gamma p}{\rho} + \frac{\mu H^2}{\rho}$

ANALYTICAL SOLUTION

For the final step, substitute the stock conditions (10) or (11) or (12) into relation (13). A first-order differential equation in ε (r) or u^2 is obtained which determines the shock. For weak shocks in the presence of a weak transverse magnetic field on substituting the stock conditions (10) in (13), neglecting the second and higher order terms of ε (since ε is very much less than unity), we find that.

$$\left(2 + \frac{\mu H^2{}_0}{\gamma p_0}\right) d\varepsilon + \left(\frac{dp_0}{p_0} + \frac{da_0}{a_0} + \frac{2dr}{r} + \frac{d\left(\mu H^2{}_0\right)}{\gamma p_0}\right)\varepsilon$$

$$= -\frac{1}{a_0^2}\left(\frac{dp}{\rho_0} + \frac{d(\mu H_0^2)}{2\rho_0} + \frac{H_0^2}{\rho_0}\frac{dr}{r} + \frac{Gmdr}{r^2}\right) \tag{14}$$

now in view of hydrostatic equilibrium prevalent in front of the shock

$$Gm\left(\frac{dr}{r^2}\right) = -\frac{dp_0}{\rho_0} - \frac{\mu H_0}{\rho_0} dH_0 - \frac{\mu H_0^2}{\rho_0}\frac{dr}{r} \tag{15}$$

Equating (14) due to (15) reduces to

$$\frac{d\varepsilon}{\varepsilon} = -1/2\left(1 - \frac{\mu H_0^2}{2\gamma p_0}\right)\left(\frac{dp_0}{p_0} + \frac{d\left(\mu H_0^2\right)}{\gamma p_0}\right)$$

$$+ \frac{da_0}{a_0} + \frac{2dr}{r} \tag{16}$$

Also we have

$$m = \frac{4\pi\rho_c}{3-w}, r^{3-w} \tag{17}$$

$$p_0 = \frac{2\pi G \rho_c^2}{(3-w)(1-w)} r^{2(1-w)} - \frac{1}{2} H_c^2 \left(\frac{1-\delta}{\delta}\right) r^{-2\delta}$$

Inserting $\delta = w - 1$ in the value of p_0 we have

$$p_0 = \left[\frac{2\pi G\rho_c^2}{(w-3)(1-w)} + \frac{1}{2}\frac{(2-w)}{(w-1)}\mu H_c^{\,2}\right] r^{2(1-w)}$$

Therefore

$$p_0 = k_1 r^{2(1-w)} \tag{18}$$

and

$$a_0 = k_2 r^{(2-w)/2} \tag{19}$$

Where

$$k_1 = \left[\frac{2\pi G\rho_c^2}{(w-3)(1-w)} + \frac{1}{2}\frac{(2-w)}{(w-1)}\mu H_c^2\right]$$

and

$$k_2 = \sqrt{\gamma k_1 / \rho_c}$$

Positivity and finiteness of the equilibrium pressure as defined by equation (18) requires that the constant w should obey the inequality

$$1 < w < 3 \tag{20}$$

Substituting dp_0 / p_0 and da_0 / a_0 in equation (16) we get

$$\frac{d\varepsilon}{\varepsilon} = \frac{1}{8}[\beta^2(2+3w)+10w-20]\frac{dr}{r} \tag{21}$$

Integrating the above equation we get

$$\varepsilon = kr[\beta^2(2+3w)+10w-20]/8 \tag{22}$$

where $\beta^2 = \dfrac{\mu H_c^2}{\gamma k_1}$ and k is constant of integration.

With the aid of equation (10), we can write

$$\frac{U}{k_2} = r^{(2-w)/2} + \frac{\gamma+1}{4} k r^{[\beta^2(2+3w)+6w-12]/8} \tag{23}$$

$$\frac{U}{a_0} = 1 + \frac{\gamma + 1}{4} k r^{\left[\beta^2 (2+3w)+10w-20\right]/8} \tag{24}$$

Similarly, for a strong magnetic field by use of shock condition (11) in (13) we get

$$\frac{d\varepsilon}{\varepsilon} = -1/2(1 - \frac{\gamma p_0}{2\mu H_0^2})(\frac{db_0}{b_0} + \frac{2dH_0}{H_0} + \frac{2dr}{r} + \frac{\gamma dp_0}{\mu H_0^2}) \tag{25}$$

Substituting the value of dp_0, dH_0 and db_0 / b_0 in this equation and integrating we get

$$\varepsilon = \bar{k} r^{\left[\left(1/\beta^2\right)\right](2+3w)+10w-20]/8} \tag{26}$$

Where $\bar{k}$ is constant of integration.
With the aid of equation (11) we can write

$$\frac{U}{k_2} = \beta\left[r^{(2-w)/2+} (3/4)\bar{k} r^{\left[\left(1/\beta^2\right)(2+3w)+6w-12\right]/8} \right] \tag{27}$$

$$\frac{U}{a_0} = b\left[1 + (3/4)\bar{k} r^{\left[(1/\beta^2)(2+3w)+10w-20\right]/8} \right] \tag{28}$$

For strong shocks, if we substitute the shock condition (12) in (13), we get

$$\left[\frac{X}{\gamma} + (1/2)(N-1)\left(\frac{X^{1/2}}{N}\right)\right] dU^2 + \left[\frac{X}{\gamma}\frac{dp_0}{p_0} - 2\left(\frac{X}{\gamma}\right)\right.$$

$$\left.\frac{da_0}{a_0} + \frac{2(N-1)X}{(N-1)+(XN)^{1/2}}\frac{dr}{r}\right] U^2 + \left[\gamma\frac{dp_0}{\rho_0} + \mu N^2 H_0 \frac{dH_0}{\rho_0} - \right.$$

$$\frac{\mu H_0^2}{\rho_0}\frac{\left[(N-1)-(XN)^{1/2}\right]N^2}{\left[(N-1)+(XN)^{1/2}\right]}\frac{dr}{r} + \frac{NGm(XN)^{1/2}}{(N-1)+(XN)^{1/2}}\frac{dr}{r^2} = 0 \tag{29}$$

Substituting the values of $da_0 / a_0, dp_0 / p_0$ and m we find that

$$\frac{dU^2}{dr} + B\frac{u^2}{r} + Cr^{1-w} = 0 \tag{30}$$

Where $\bar{B} = AB, \bar{C} = AC, A = \left[\frac{X}{\gamma} + (1/2)(N-1)\left(\frac{X^{1/2}}{N}\right)\right]$

$$\bar{B} = \left[\frac{2(N-1)\gamma}{(N-1)+(XN)^{1/2}} - w\right](x/\gamma)$$

$$\bar{C} = \left[\frac{2k_1\gamma(1-w)}{\rho_c} + \frac{\mu N^2 H_c^{\,2}(1-w)}{\rho_c} - \frac{\mu H_c^2\left[(N-1)-(XN)^{1/2}\right]N^2}{\rho_c\left[(N-1)+(XN)^{1/2}\right]}\right.$$
$$\left. + \frac{NG(XN)^{1/2}4\pi\rho_c}{[(N-1)+(XN)^{1/2}](3-w)}\right]$$

On integration (30) disclose that

$$U^2 = \bar{\bar{K}} r^{-B} - \frac{Cr^{(2-w)}}{B+(2-w)} \tag{31}$$

Where $\bar{\bar{K}}$ is constant of integration.

The shock strength can be written as

$$\left(\frac{U}{a_0}\right)^2 = \frac{\bar{\bar{k}}}{k_2^2} r^{w-(B+2)} - \frac{C}{k_2^2(B+2-w)} \tag{32}$$

Finally, the expression for the pressure, the density and the particle velocity just behind the shock for above cases can be expressed as

Weak shock with weak magnetic field :

$$p = k_1 r^{2^{(1-\varpi)} + \gamma k_1 kr\left[\beta^2(2+3w)+2w-4\right]/8} \tag{33}$$

$$\rho = \rho_{cr^{-w} + \rho_c kr\left[\beta^2(2+3w)+2w-20\right]/8}$$

$$u = kk_2 r^{\left[\beta^2(2+3w)+6w-12\right]/8}$$

Weak shock with strong magnetic field :

$$p = k_1 r^{2(1-\varpi)} + \gamma k_1 \bar{k} r^{\left[1/\beta^2(2+3w)+6w-12\right]/8}$$

$$\rho = \rho_{cr^{-w}} + \rho_c \overline{kr}^{\left[(1/\beta^2)(2+3w)+2w-20\right]/8} \tag{34}$$

Strong shock with srong magnetic field :

$$p = \frac{k_1}{k_2} X(N)\left[\bar{\bar{k}} r^{-(w+B)} - \frac{C r^{2(1-w)}}{(B+2-w)}\right] + Y(N) k_1 r^{2(1-w)}$$

$$p = (1-1/N)(\bar{\bar{k}} r^{-B} - \frac{C}{(B+2-w)} r^{2-w}) \qquad (35)$$

RESULTS AND DISCUSSION

The expression (23) and (27), in general, represent the propagation of weak diverging spherical shock through a self-gravitating gas in the presence of weak and strong magnetic field, respectively.

For $\frac{U}{a_0} = 1.25$ at $r = 100$; the integration constant k and $\bar{k}$ for weak shocks in the presence of weak and strong magnetic field are given in tables 1 and 2 for w = 2 and 2.5. Similarily for strong shock with strong magnetic field, when U/ a_0 = 20 at r = 100, the calculated values of c/k_2^2 and $\bar{k}/k_2^2$ are given in table 3 for w = 1.75, 2.0 and 2.5. Now calculation can be done easily. The conclusion is that the effect of the magnetic field (β^2) is significant on the flow pattern.

TABLE-1

Weak shock with weak magnetic field

β^2	W	K
0.01	2.00	0.3979135
0.10		0.2628988
0.25		0.1317615
0.01	2.50	0.0221839
0.10		0.0135609
0.25		0.0059708

TABLE-2

Weak shock with weak magnetic field

β^2	W	$\bar{k}$
1.75	2.00	$-5.2862155 \times 10^{-3}$
2.00		-0.0154822
2.50		-0.0442566
1.75	2.50	- 0.057393
2.00		-0.1787856
2.50		-0.5571584

TABLE-3

Strong shock with strong magnetic field N=5, = 1.4

β^2	W	$\frac{\bar{k}}{k_2^2}$	$\frac{C}{k_2^2}$	B
2.00	-	792.39106	-58.633031	-
	1.75			0.0465122
2.50		664.06213	-68.345757	
2.00		437.88152	-53.914952	
	2.00			-0.1326276
2.50		459.00189	-59.0754251	
2.00		15.666711	-105.41187	
	2.50			-0.2676282
2.50		18.093003	-169.28974	

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DERIVATION OF A GENERAL CLASS OF BILATERAL FUNCTIONS FOR LAGUERRE POLYNOMIALS OF TWO VARIABLES

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ABSTRACT

A new class of bilateral generating functions for the Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ have been obtained.

Mathematics Subject Classification (2000) : 33C 45

INTRODUCTION

Applying Lie-group theoretic of W. Miller, we obtain the following operator for $L_n^{(\alpha,\beta)}(x,y)$ by giving suitable interpretation to *n*:

$$R=e^{z}\left\{x\frac{\partial}{\partial x}+\frac{\partial}{\partial z}+(\alpha+1-x)\right\} \tag{1}$$

such that

$$R\left[L_n^{(\alpha,\beta)}(x,y)t^n\right]=\frac{(n+1)^2}{\beta+n+1}L_{n+1}^{(\alpha,\beta)}(x,y)t^{n+1}, \text{where } e^z=t \tag{2}$$

The extended form of transformation group generated by R is

$$e^{WR}\left[f(x,y,t)\right]=(1-wt)^{-\alpha-1}e^{wxt/(wt-1)}.f\left[\frac{x.}{1-wt},y,\frac{t.}{1-wt}\right] \tag{3}$$

where $f(x, y, t)$ is an arbitrary differentiable function.

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The object of the present is to derive a general class of bilateral generating function by employing group-theoretic method. Actually our result can be put in the form of a theorem as follows :

THEOREM

If there exists a unilateral generating functions of the form

$$G(x,y,t) = \sum_{n=0} a_n L_n^{(\alpha,\beta)}(x,y)t^n \tag{4}$$

then the following of bilateral generating functions will hold :

$$(1-t)^{-\alpha-1} e^{xt/(t-1)} G\left[\frac{x}{1-t}, y, \frac{tz.}{1-t}\right]$$

$$= \sum_{p=0}^{\alpha} \sum_{n=0}^{p} \frac{a_n}{(p-n)!} \frac{(p!)^2(\beta+n)!}{(n!)^2(\beta+p)!} L_n^{(\alpha,\beta)}(x,y) z^{n+p} t^n \tag{5}$$

Importance of the result of Eq. 5 is that whenever one knows a generating functions of the type given in Eq. 4 for a particular value of a then the corresponding bilateral generating functions can at once be written down from Eq. 5. Thus, one can derive a large number of bilateral generating functions by setting different values to a_n.

DERIVATION OF GENERATING FUNCTION

Let

$$G(x,y,t) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha,\beta)}(x,y)t^n \tag{6}$$

Now replacing *t* by *tz*, we have

$$G(x,y,tz) = \sum^{\alpha} a_n L_n^{(\alpha,\beta)}(x,y)t^n z^n$$

We operate both sides by exp (wR)

$$\exp(wR)\, G(x, y, tz) = \exp(wR) \sum_{n=0}^{\infty} a_n L_n^{(\alpha,\beta)}(x, y) t^n z^n \tag{7}$$

Left hand side of Eq. 7 becomes

$$\text{(A)} \quad (1 - wt)^{-\alpha-1} e^{wxt/(wt-1)} G\left[\frac{x}{1 - wt}, y, \frac{tz}{1 - wt}\right]$$

On the other hand, right hand side of Eq. 7 reduces to

$$\text{(B)} \quad \sum_{p=0}^{\alpha} \sum_{n=0}^{p} \frac{a_n w^{p-n}}{(p-n)!} \frac{(p!)^2 (\beta + n)!}{(n!)^2 (\beta + p)!} L_{n+p}^{(\alpha,\beta)}(x, y) z^{n+p} t^n$$

Equating (A) and (B) and substituting $w = 1$, we obtain Eq. 5.

PARTICULAR CASE

Setting $y = 0$ and $\beta = 0$, we have the following bilateral generating function for $L_n^{\alpha}(x)$:

If

$$G(x, t) = \sum_{n=0}^{\alpha} a_n L_n^{a}(x) t^n$$

then the following bilateral generating function will exist

$$(1 - t)^{-\alpha-1} e^{xt/(t-1)} G\left(\frac{x}{1 - t}, \frac{tz}{1 - t}\right) = \sum_{p=0}^{\infty} \sum_{n=0}^{p} a_n \binom{p}{n} L_p^{\alpha}(x) z^p t^n$$

This was derived by *Al salam*.

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ANALYSIS OF TRAFFIC NOISE PARAMETERS IN RELATION WITH PHYSIOLOGICAL EFFECTS

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ABSTRACT

Disturbance caused by noise is one of the most important health consequence. Over a number of years, investigations have shown that noise affects different activities and becomes the causative factor of environmental pollution. In view of the rapid infrastructural developments it is essential to study the traffic noise with respect to various causative factors (noise 65>dB leads to the annoyance i.e. above the permissible range). Keeping in view of the alarming increase of noise due to heavy traffic, engine sounds and blowing the horns, it is necessary to approximate the noise parameters. Present study takes into account the noise measurements on NH-58 at different noisy locations. Data is recorded using Noise Analyzer B&K2260 with software BZ7815.

Keywords : Noise, traffic, sleep disturbances.

INTRODUCTION

Sounds are pressure waves travelling through air (or through fluids and solids). The waves, which are produced from the source point, will result in mechanical disturbances in material medium. Pressure variations propagate away from place of origin by transfer of energy from molecules are well associated with the pollution of dust particles which include chemicals (like carbon monoxide, nitrogen, hydrogen etc.). Sound pressure waves produced due to diesel generator vehicles moving on the road release the smoke and the sound (the engine sound and the horn sound) are the added parameters to the environmental pollution. Ultimately, the molecules, which carry quantum of these unwanted waves, cause the hazard on the human beings health. We notice the damage of hearing due to periodic compactions and rarefaction in the eardrum on tympanic membrane, the reduce of work efficiency the arrest of respiration due to the large quantity of carbon and dust particles enter the lungs and spech

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interference within the communication and other related problems. The regulation prohibiting the sounding of motor horn in a built-up area on any road is specified for the fixed time duration.

Ahrlin and Ohrstrom [1] made a review of different kinds of noise exposure effects in relation to medical effects. Talbott et al. [6] focused on the study of occupational noise exposure for the increase of blood pressure levels. Van Dijk [7] reviewed the non-auditory effects of noise in industry comparing to stress factors. Griefahn [3] investigated the cardiac responses due to military noise at the time of sleep. Makarewicz [4] gave the model for A-weighted sound exposure level in a built-up area by investigating excess attenuation. Yoshida [9] experimentally demonstrated that noise at low or moderate levels could affect performance. Watts [8] considered the study of traffic running on concrete and bituminous road surfacing. Omiya et al. [5] examined the noise dose relationship and ratings of environmental noise by L_{50} and L_{eq} with the help of stored data and concluded that noise environments irrespective of time of day and land use. Ando et al. [2] formulated for subjective attributes for noise fields based on the model of the auditory brain system, the sensitivity of human ear to sound source formed by physical system being compared with response of the cochlea.

In this paper a study to approximate the noise parameters due to traffic noise measurements in relation to physiological effects when human beings are exposed to traffic noise for a longer period.

FORMULATION

Sound pressure waves have been propagated as plane waves with the quantity velocity, time, displacements, frequencies and the selected noise parameters. The noise can be assume to the unidirectional motion from the source point to the receiving point, the level of the noise varies in terms of various noise producing vehicles such as : noise from the engine, noise from the speed and noise by blowing the horns. Taking x as the displacement, t as the time, u as the velocity and c as the reference velocity,

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$U_n = c^2 \nabla^2 u(x,t) \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{3}$$

Equation (3) is employed in equation (2) for one-dimensional system. The ranges of 'x' and 't' are considered at three different locations and timings for monitoring the noise measurements. Measurements are repeated in different time steps also. Distances for monitoring are taken at 5, 10 and 15 meters from the center (noise source point) of the road. The effective value of the noise monitoring is the best measure for continuous sound at short and long duration. This assessment is characterized by a parameter called crest factor C_f which is the peak sound pressure level to the RMS value of the sound. It is approximated as the logarithmic ratio of peak levels and RMS values with reference to the sound wave.

$$C_f = K_1[K_{11} - K_{22}] \tag{4}$$

where, $K_1 = 20,\ K_{11} = \log p_{peak},\ K_{22} = \log p_{rms}$

ANALYSIS

Solving the equation (1) by the closed form technique (tentative solution with specification of variables) as

$$u(x,t) = X(x)T(t) \tag{5}$$

where, X is a function of x alone.
T is a function of t alone.

Introducing (5) in (1) we get after discritization

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = k \tag{6}$$

The value of k is for three different variations $k > 0,\ k = 0,\ k < 0$.

From equation (6) taking separate differential equations in t and x respectively and solving them we obtain the solutions with successive arbitrary constants. The solutions for $k > 0$ and $k = 0$ do not represent the physical nature of the problem. Since the terms with $k < 0$ involve the way parameters. Therefore, the solution of (1) for $k < 0$ is

$$u(x,t) = \left\{A\cos\sqrt{kt} + B\sin\sqrt{kt}\right\}\left\{C\cos\left(\frac{\sqrt{k}}{c}\right)x + D\sin\left(\frac{\sqrt{k}}{c}\right)x\right\} \tag{7}$$

The constants A, B, C, D have been evaluated by prescribing the boundary conditions with variations of displacements x from 2 to 15 meters and time t from 1 to 30 hours. The term $\sqrt{k}$ is replaced by $2\pi f$ per second (the frequency function which takes the noise measurements for showing the values of L_{eq}. Values of $u(x,t)$ have been calculated for the noise measurements taken at three locations (shown in the table 1). Noise measurements have been recorded at three locations for different types of vehicle sound and the noise values have been employed in equation (7).

In terms of energy emission levels from the source point to the receiving point it is necessary to reduce the noise levels in terms of various adjustments such as allowing the vehicles in the lanes adjusting the reaching distance alternately, the barriers with different heights.

Traffic flow adjustment (ΔT)

$$\Delta T = 10\log(V_i\pi D_0 / S_i T) \tag{8}$$

It is necessary to analysis the adjustment of traffic flow in order to reduce the noise effect in term of distances for vehicle into the traffic stream. This is accomplished through the application of the traffic flow adjustment. Three of the five subparameters in this adjustment are constants, with π representing an integration constant for the assumed infinite roadway segment (acoustic angle of influence is π, or 180 degree). D_0 represents the reference distance of 10m, and T, the time period, is 1 hr.

$$10\log 2V_i = 10(0.3) + 10\log V_i \tag{9}$$

Conversely, if speed was doubled and everything else held constant, the result would be a 3-dB decrease in Leq.

$$10\log\frac{1}{2S_i} = 10\log\frac{1}{2} + 10\log\frac{1}{S_i}$$

$$= -3 + 10\log\frac{1}{S_i} \tag{10}$$

The decrease in ΔT is that, with volume held constant doubling the vehicular speed actually the amount of time that each vehicle acoustically influences a given receiver of sound.

Distance adjustment $(\Delta \mathbf{D})$

$$\Delta D = 10\log(D_0 / D)$$

The distance adjustment or spatial decay rate can be established to be 3 dB per distance doubling, if D is doubled.

$$10\log\frac{1}{2D} = 10\log\frac{1}{2} + 10\log\frac{1}{D}$$

$$= -3 + 10\log\frac{1}{D} \tag{11}$$

Barrier adjustment (ΔB)

The placement of an obstruction between the highway and the observer will shield the observer.

In most instances where the shielding adjustment is used, solid barriers or walls are involved. Barrier attenuation analytically defines the amount of acoustic energy loss encountered when sound rays are required to travel over and around a barrier.

The Fresnel number N may be calculated as

$$N = 2\frac{\delta}{\lambda} \tag{12}$$

where, δ is the path length difference, λ wavelength.

Thus, a barrier with $\delta = 0.9144m$ would have $N = 273$.

RESULTS AND DISCUSSION

The elevated noise measurements measured at noisy locations describe the traffic noise effects on various geometrical dimensions keeping in view that the allowable range is 65 *dB*. The Frequency range from 800 to 4000 hertz is employed in the analysis. It is clear that the road traffic noise data measured disagree with the allowable range. These elevated values of noise ensure the sleep disturbance, activity interference for the human beings living close to the noise source. Also the prolonged exposure with these elevated noise cause the hearing loss and damage on the hearing system which becomes the significant physiological effect. The angular velocity of the sound wave varies from the allowable (300 m/sec) to the elevated level (i.e.>400 m/sec) justifies the corresponding increase in the normal velocity. From equation (7), the increase of velocity by 11.5184 m/sec induce the temporary threshold shift at frequency 3500 Hz and further increase of velocity by 13.90696 m/sec will become the cause for the effective threshold shift indicating physiological destruction within in the hearing mechanism. The higher range of crest factor C_f46.1-52.26 dB at 5 to 15 m show the best measures (equation 4) to analysis the short and long sound waves. As the distance from the source increase there is a decrease in crest factor, this shows that as the distance from the source increases, effect ofnoise diminishes. It can be concluded that some noise control techniques may be employed at noise source transmission path and at the receiver. At source, reducing the exciting forces and changes in operating procedure can control noise. At the transmission path, it can be controlled by the construction of the barrier (equation 8-12) and absorption devices. The receiver can use the high quality protective equipment in order to avoid consequences of the noise.

Table 1 : Hourly Variation of Leq.

Location 1		Location 2		Location 3	
Time	Leq	Time	Leq	Time	Leq
10-11	80.2	10-11	82.3	10-11	77.3
11-12	83.1	11-12	81.6	11-12	77.7
12-1	82.2	12-1	81.5	12-1	79.1
1-2	82.3	1-2	82.6	1-2	77.5
2-3	82.2	2-3	81.8	2-3	78.6
3-4	78.7	3-4	81.9	3-4	78.0
4-5	78.9	4-5	81.7	4-5	77.9
5-6	82.1	5-6	82.4	5-6	79.2

Table 2 : Calculation of Crest Factor (C_f).

Displacement	LA_{eq} (peak)	C_f	C_f (Average)
5 meters	105.0	52.474	52.56
	105.1	52.472	
	105.1	52.708	
	97.5	45.656	
10 meters	102.2	47.467	49.69
	102.2	49.218	
	103.6	50.200	
	103.6	49.957	
15 meters	102.3	50.445	46.10
	97.9	53.452	
	97.9	44.461	
	98.1	46.510	

Table 3 : Classified Traffic Volume at Various Locations.

Location 1

Time	C/J/V*	Scooter	M/C**	Bus	Truck	Tempo	Tractor
10 to 11	205	193	107	81	46	44	25
11 to 12	187	203	97	87	37	38	22
12 to 1	193	249	84	80	32	34	36
1 to 2	275	369	132	86	42	11	19
2 to 3	175	187	104	80	41	20	19
3 to 4	173	97	59	79	42	17	10
4 to 5	177	130	40	61	41	26	19
5 to 6	181	175	111	85	47	31	10

Location 2

Time	C/J/V	Scooter	M/C	Bus	Truck	Tempo	Tractor
10 to 11	211	163	120	154	79	112	67
11 to 12	191	154	99	157	88	122	57
12 to 1	201	140	110	120	61	75	26
1 to 2	214	146	52	126	63	61	37
2 to 3	140	201	88	128	72	68	40
3 to 4	177	212	113	105	51	90	39
4 to 5	270	336	105	120	102	78	27
5 to 6	245	259	117	111	92	81	21

Location 3

Time	C/J/V*	Scooter	M/C**	Bus	Truck	Tempo	Tractor
10 to 11	190	165	74	41	65	87	40
11 to 12	168	196	80	35	72	88	44
12 to 1	156	100	45	56	21	88	32
1 to 2	122	58	36	42	44	51	31
2 to 3	154	71	38	64	58	66	27
3 to 4	174	125	66	36	78	81	35
4 to 5	182	142	84	76	70	81	39
5 to 6	176	121	90	56	65	82	25

* C/J/V - Car/Jeep/Van
** M/C - Motor Cycle

Table 4 : Individual Vehicle Noise Levels.

Vehicle Type	Car	Jeep	Van	Scooter	Motor Cycle	Tempo	Bus	Truck	Tractor
Noise Level (dB)	76.14	78.01	77.80	75.10	76.55	84.14	86.1	87.30	89.9

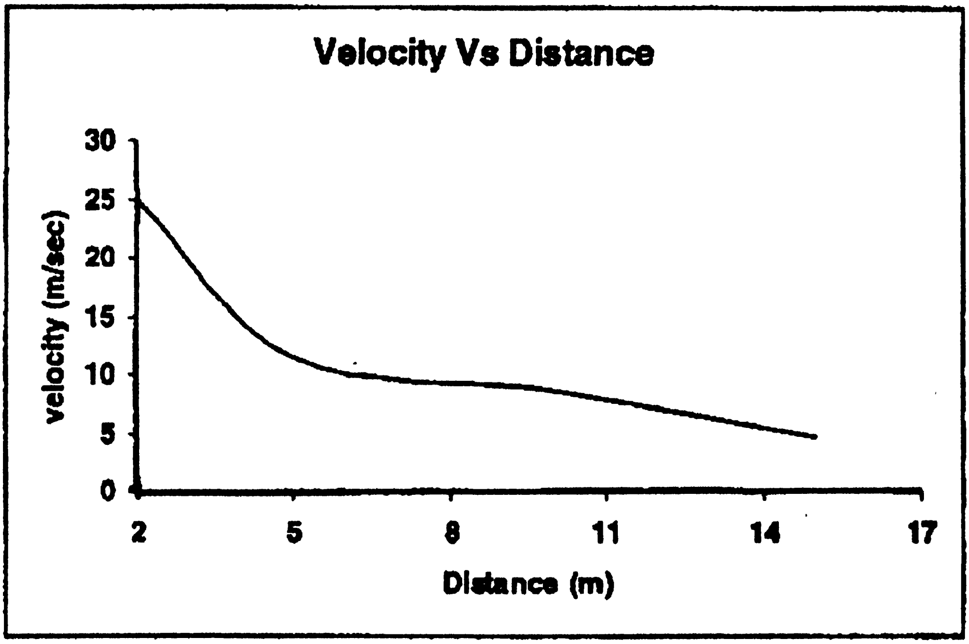

Fig. 1 : Velocity Vs Distance

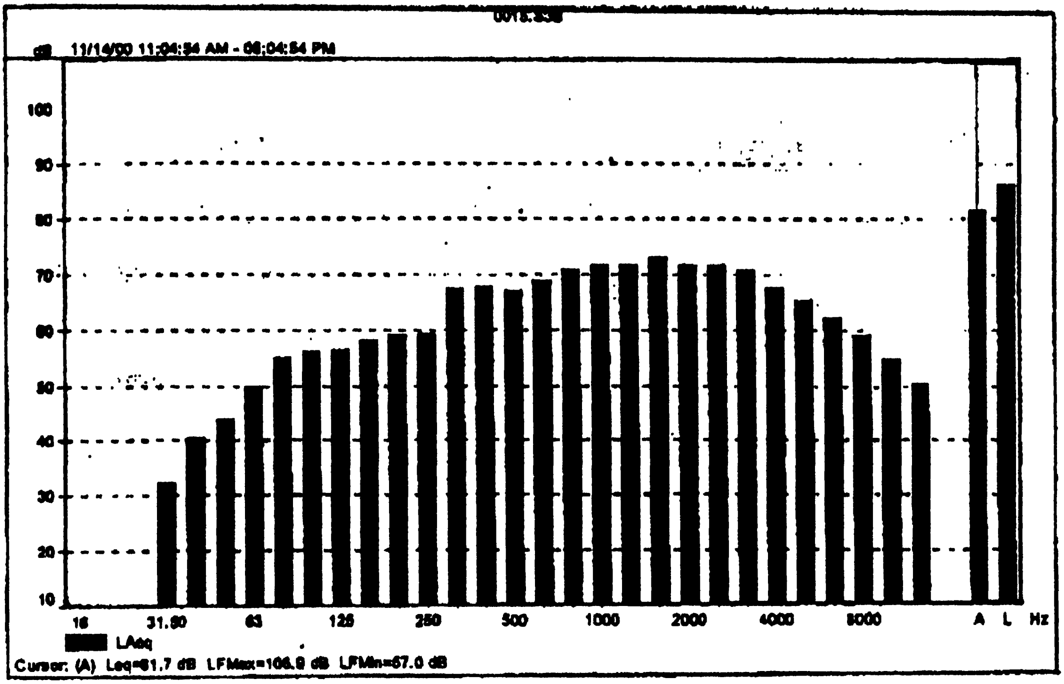

Fig. 2 : Spectrum Showing Leq

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WHEN IS A FUNCTION CONTINUOUS ?

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ABSTRACT

In this note, we give three decompositions of continuity which are not widely known. For example every text-book in calculus contains following: (a) A continuous function preserves compact sets. (b) A continuous function preserves connected sets (the intermediate value theorem). That (a) and (b) together give a nice space is not found in text-books.

Mathematics Subject Classifications (2000): 54 C 05, 54C08.
Keywords and Phrases: Continuity, connected function, compactness preserving function, decompositions.

1. INTRODUCTION

In the mathematical literature there are concepts and results which are simple, yet profound and beautiful. Some of them can be understood by undergraduates but are not widely known. In this note it is proposed to present a few gems related to continuity. The results are from elementary analysis and have successive generalizations to metric spaces, topological spaces, and topological vector spaces. In order to make this note tangiable, we give proofs for functions on IR to IR. Then in the remarks we suggest how the generalizations are conceived. We provide a reasonably complete bibliography which will be useful to those who wish to pursue the subject further.

The main theme in the present note is the decomposition of continuity into two weaker conditions. The classical case that continuity is equivalent to upper and lower semi-continuity is widely known. Here we present three other such decompositions. The concept of continuity being one of the most important concepts in mathematics, any such decomposition will naturally shed more light on it.

2. COMPACT AND CONNECTED

Suppose $f: IR \to IR$ is continuous. The following results are well known :

(a) f takes compact sets into compact sets, and
(b) f takes connected sets into connected sets (Intermediate Value Theorem).

But it is not widely known that (a) and (b) together imply that f is continuous! (See [10, 17]).

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(2.1) THEOREM. If $f: IR \to IR$ satisfies (a) and (b) then it is continuous.

Proof : It is sufficient to show that f is continuous at 0. Suppose it is not. Let $U_n = \left(-n^{-1},\, n^{-1}\right)$. Then there is a closed interval B_0 with $f(x)$ as midpoint such that for each $n \in IN$, $f(U_n) \not\subset B_0$. Thus there is an $x_1 \in U_1$ such that $f(x_1) \not\in B_0$. Hence there is a closed interval B_1 with midpoint $f(x_1)$ and disjoint from B_0. Since f preserves connected sets, $f(U_2) \not\subset B_0 \cup B_1$ and so there is an $x_2 \in U_2$ such that $f(x_2) \not\in B_0 \cup B_1$. Then there is a closed interval B_2 with midpoint $f(x_2)$ and disjoint from $B_0 \cup B_1$. Inductively we have $x_n \in U_n$ with $f(x_n) \not\in \cup\{B_k{:}0 \le k \le n-1\}, B_k$ a closed interval with midpoint $f(x_k)$ and $\{B_k{:}k \in IN\}$ pairwise disjoint. Clearly $x_k \to 0$. So the set $E = \{0\} \cup \{x_k{:}k \in IN\}$ is compact but $f(E)$ is not compact, a contradiction of (a).

(2.2) REMARK. Suppose X and Y are metric spaces and $f: X \to Y$ satisfies (a) and (b). Then, noting that in the above proof each U_n is connected, we find that f is continuous if X is locally connected ([10]). This result was extended in [WH] to topological spaces where X is locally connected and first countable and Y regular. The paper contains counter examples to illustrate the need for these conditions. After about a decade, it was shown in [7] that White's result is true if Y is merely Hausdorff rather than regular.

3. CONNECTED AND CLOSED FIBERS

If $f: IR \to IR$ is continuous, then for each $y \in IR$, the fiber $f^{-1}(y)$ closed and, of course, f preserves connected sets. The following converse was proved in [16] :

(3.1) THEOREM. Suppose $f: IR \to IR$ takes connected sets into connected sets and for each $y \in IR$, the fiber $f^{-1}(y)$ is closed. Then f is continuous.

Proof : Suppose $x \in IR$ and $f(x) \in (c,d)$. Then $f^{-1}\{c,d\}$ is a closed subset not containing x. So there is an open interval I containing x and disjoint from $f^{-1}\{c,d\}$. Since f(I) is connected and contains $f(x), f(I) \subset (c,d)$. Thus f is continuous.

(3.2) REMARKS. To understand how this result can be generalized, we note the following:

(a) Open intervals are connected and form a base for the topology of IR.

(b) The set $\{c,d\}$ is the boundary of the open interval (c,d).

Rowe's result (3.1) was successively generalized to metric spaces [10], to Hausdorff topological spaces [12] and finally to a topological result with applications to Functional Analysis [8]. We give below Hrycay's result whose proof is as easy as that of (3.1):

(3.3) THEOREM. Suppose X is a topological space with a base β of connected open sets and Y is a regular topologicl space with a base β' of open sets. Suppose $f: X \to Y$ is a function satisfying:

(a) for each $B \in \beta$, $f(cl\, B)$ is connected, (i.e. f is connected-β), and

(b) for each $B' \in \beta'$, $f^{-1}[\partial B']$ is closed. ($\partial B'$ is the boundary of B'.)

Then f is continuous. (See [8].)

The result (3.3) includes, as special cases, most of the previous results of this type and moreover, has applications to Functional Analysis. In fact, Hrycay arrived at this result in an attempt to solve a problem in Functional Analysis. In 1966, I was teaching a course in Functional Analysis and also conducting a seminar in connected and other non-continuous functions. In the course we studied the result:

(3.4) A Banach space X is finite dimensional if any and if every linear functional on it is continuous.

It is reasonable to conjecture that if the Banach space X is infinite dimensional, then any linear functional on X, although not continuous, ought to satisfy some weaker property implied by continuity, such as preservation of compactness or connectedness. So I made the following conjecture:

(3.5) CONJECTURE. Every linear functional on a Banach space preserves connected sets.

Hrycay who was looking for a topic for his Thesis, took up this problem and showed the following:

(3.6) For every topological vector space (TVS) X, there is a base β of connected open sets such that every linear functional on X is connected-β (see (3.3) (a).)

From (3.3) and (3.6) he deduced the well known result: a linear functional on a TVS is continuous if and only if the fibers are closed.

(3.7) A linear functional on a TVS is continuous if and only if it takes connected sets into connected sets.

Thus the conjecture (3.5) is false. Hrycay's work showed the significance of the generalization from connected functions to functions that are connected-β. For details see [8].

4. QUASI AND NEAR CONTINUITY

Two weaker forms of continuity have been in the literature for over half a century and are widely known to workers in Topology and Functional Analysis. Before describing them, let us recall that

(4.1) a function $f: X \to Y$ is continuous at $x \in X$ if for each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $U \subset f^{-1}(V)$.

By changing the condition "$U \subset f^{-1}(V)$" in different ways, we get weaker forms of continuity.

(4.2) DEFINITION. A function $f: X \to Y$ is nearly continuous at $x \in X$ if for each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $U \subset cl f^{-1}(V)$.

We say that f is nearly continuous if it is nearly continuous at each $x \in X$.

Obviously a continuous function is nearly continuous and the following well known example shows that the converse is not true : $g: IR \to IR$ given by

$$g(x) = 0 \text{ when } x \text{ is rational, and}$$
$$= 1 \text{ when } x \text{ is irrational.}$$

(4.3) REMARKS. Blumberg [2] discovered nearly continuous functions which he called "densely approaching", certainly a more appropriate term. Some authors have called such functions "almost continuous" which caused some confusion since some other weaker forms of continuous functions are also known under this term. The importance of near continuity stems from the result :

(4.4) Every linear function from one Banach space to another is nearly continuous.

Thus near continuity and its dual, near openness, form intermediate steps in the well known closed graph and open-mapping theorems in Functional Analysis. Moreover, Blumberg proved a spectacular result :

(4.5) Every function $f: IR \to IR$ is nearly continuous on a dense subset of IR. (i.e. f has the Blumberg property).

There is a vast literature on the Blumberg property (see [13]).

(4.6) DEFINITION. A function $f: X \rightarrow Y$ is said to be quasi-continuous at $x \in X$ if for each neighbourhood U of x and each neighbourhood V of $f(x)$, there is a non-empty open set $W \subset U \cap f^{-1}(V)$. A quasi-continuous function is one that is quasi-continuous at each $x \in X$.

Clearly if W contains x, then f is continuous and so continuity implies quasi-continuity. That the converse is not true is given by the following example: $h: IR \rightarrow IR$ given by:

$h(x) = \sin(x^{-1})$ when $x \neq 0$, and $h(x) = 0$ when $x = 0$.

(4.7) REMARKS. The following is a well known example of a function $s: IR \times IR \rightarrow IR$ that is separately continuous but not jointly continuous:

$$\begin{aligned} s(x,y) &= xy(x^2+y^2)^{-1} && \text{when } (x,y) \neq (0,0) \\ &= 0 && \text{when } (x,y) = (0,0) \end{aligned}$$

It was first observed by Volterra, who was quoted by Baire [1], that

(a) every separately continuous function $f: IR \times IR \rightarrow IR$ is quasi-continuous. Moreover,
(b) every such quasi-continuous function is continuos on a residual subset of $IR \times IR$.

From (a) and (b) follows a classical result due to Baire that separate continuity of $f: IR \times IR \rightarrow IR$ implies the joint continuity of f on a residual subset of $IR \times IR$. Thus quasi-continuity serves as an important intermediate step in the proof of Baire's result. There is a vast literature on this topic of separate versus joint continuity (see 11, 14, 15). Quasi-continuity was systematically sudied by Kempisty [9] and has been rediscovered [3].

Although these two important generalizations of continuity viz. near and quasi continuity have been studied extensively for a long time, it was only relatively recently recognized that they give a nice decomposition of continuity (see [4, 13]).

(4.8) THEOREM. If $f: IR \rightarrow IR$ is nearly continuous and quasi-continuous, then it is continuous.

Proof : Suppose f is not continuous at $x \in IR$. Then there is a closed interval B, with midpoint $f(x)$, such that in every open interval I containing x, there is a y with $f(y) \notin B$. Since f is quasi-continuous at y, there is an open set $W \subset I$ such that $f(W) \subset B^c$. Then $I \not\subset cl\, f^{-1}(B)$ which contradicts the fact that f is nearly continuous.

(4.9) REMARKS. The above result is a *global* one and does not hold locally. For example the function $j: IR \rightarrow IR$ given by $j(x) = \sin(x^{-1})$ when x is rational, and

$j(x)=0$ when x is irrational is nearly and quasi continuous at 0 but is discontinuous every where. In case of topological spaces, a function that is nearly and quasi continuous is θ – continuous. A function $f: X \rightarrow Y$ is θ – continuous iff for each $x \in X$, and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset clV$. Clearly if Y is regular, then θ – continuity is equivalent to continuity. We have discussed here three decompositions of continuity. There are others; See e.g. [5, 6].

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A HISTORY OF DETERMINING VALUES OF π FROM ANCIENT TO MODERN PERIOD

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ABSTRACT

In this paper values of Pie determined by various mathematicians from ancient to modern period are given. Also we present a number of series representation for Pie.

Keywords : Tritah; Circumference; Diameter; Circle.
AMS Classification Number : 01A32

Many mathematicians studied the values of Pie from very beginning. Some of them are *Āryabhaṭa I* (b. 476. A.D.) *Brahamagupta* (b. 598 A.D.), *Mahāviracārya* (b.629 A.D.) *Bhāskaracārya, Gaṇes'a Daivajña* (b.1507 A.D.) [1-16, 20, 23, 24, 27].

The oldest value of Pie is found in Ṛgveda (2500 B.C.) (for instance, [8, 24]) According to Ṛgveda:
Bhinad Valasya Paridhīn iva tritaḥ (Ṛgveda 1-52, 5; [8], p.187)
This means
The ratio of circumference (*paridhi*) and diameter (*vyās*) is 3:1 and it is called *tritaḥ*.
That is,

$$\frac{c}{d} = \frac{3}{1} = tritaḥ$$

Where c and d stand for circumference and diameter of a circle respectively.

Baudhāyana (1000 B.C.) suggested that value of Pie is equal to 3 [39].

Archimedes (b.287 B.C.) had shown the Pie to be the ratio between circumference of a circle and its diameter, and found that it lies between $\frac{223}{71}$ and $\frac{220}{70}$ [18-20, 23]. *Archimedes* was the first who introduced the notation π [40].

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After *Archimedes, Liz Hui* (Chinese mathematician(c.263 A.D.)) calculated the gross value of Pie = 157/50 = 3.14 and near value 3927/1250 = 3.1416 [23]. *Tsu-Chung-Lhih* (b. 470 A.D.) in *Tru Su* gave the value of Pie equal to 355/113 [23, 24]. For the details see: Ancient Chinese Technology and Science, pp. 52-123.

For detailed sources on the simplest approximation 3, refer to [26]. These include sources such as Jaina, Buddhistic, Christian, Chinese, apart from Hindu sources. Mishra and Singh [26] also present a good account of the values $\frac{22}{7}, \sqrt{10}, \frac{355}{113}, \frac{600}{191}, \frac{16}{5}$ and $\left(\frac{16}{9}\right)^2$ for Pie in almost all culture areas of the world. Indian knowledge for Pie is further enriched by [25] where authors establish *Virasena*'s knowledge for $\frac{355}{113} > \pi > \frac{62832}{20000}$ through first degree indeterminate analysis concept in the equation $c = 3d + \frac{(16d+16)}{113}$ of *Virasena* (a ninth century A.D. Jaina mathematician).

Āryabhaṭa I (b. 476 A.D.) was the first Indian mathematician who gave the value of Pie in his *Āryabhatiam* (c. 499 A.D.) correct to four decimal places [6, 10, 15, 21, 23, 24, 28, 34]. He says the following verses:

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्त्राणाम्
अयुतद्वयधिष्कम्भस्यासन्नो वृत्तपरिणाहः। ([24], p.28)

This means

100 plus 4 multiplied by 8 and added to 62000, this is approximate measure of circumference of circle whose diameter is 20000.

$$\frac{c}{d} = \frac{(100+4).8+62000}{20000} = \frac{62832}{20000} = 3.1416$$

After *Āryabhaṭa* I, *Brahmagupta* and *Mahāviracārya* considering many figures concluded that circumference divided by diameter is 3 or square root of 10 [15, 21, 23, 34]. In other words, for concluding the value of Pie they could not be as accurate as their predecessors.

Bhāskaracārya II (b. 1114 A.D.) gave the formula in his *Līlavāti* for Pie [1, 3, 6, 9, 11, 22, 23, 27] in the following stanza.

व्यासे भनन्दाग्नि हते विभक्ते खबाणसूर्यैः परिधिस्तु सूक्ष्मः ।
द्वाविंशति घ्ने विहृतेऽथ शैलेः स्थूलोऽथवा स्याद्व्यवहार योग्य ।

([27], p.143)

This means

Diameter of a circle multiplied by 3927 and divided by 1250 gives the near circumerence, alternatively, the diameter multiplied by 22 and divided by 7 gives the gross value. Near value of Pie is equal to $\frac{3927}{1250}$ and gross value $\frac{22}{7}$.

After *Bhāskaracārya II* majority of mathematical work was in progress in southern part of India, and Kerala's mathematicians determined the series for Pie.

Patuman Somayāji in his *Karaṇpaddhati* (c.1438 A.D.) gave the series [28, 35]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - - - - - - - \infty$$

In *Tantra Sangraha* (c. 1502 A.D.), *Nīlkaṇṭha* had been anticipated this series before *Gregory*.

After two hundred fifty years this very series was also enunciated by European *James Gregory* (1638-1675 A.D.) in 1671 [23, 24, 28].

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - - - - - - - \infty$$

If we take $x = 1$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - - - - - - - - - \infty$$

or $$c = 4d - \frac{4d}{3} + \frac{4d}{5} - - - - - - - - \infty$$

From this series Pie can be found for ten decimal places. i.e.

$\pi = 3.1415926536$

The Kerala School provided many approximations to Pie. One such value of Pie is given in *Kaṭapayādi* notation [28, p. 177] is that the circumference is represented by

चण्डाशुचन्द्राधमकुम्भिपालः

and diameter by

आनूननूत्राननुत्रनित्यम् ।

Then the good approximation of Pie $= \dfrac{31415926536}{10{,}000{,}000{,}000}$

$= 3.1415926536$ which is correct to ten decimal places

Mādhava (1340-1425 A.D.) gave rational approximation for Pie by considering finite number of terms [11, 12, 15-17, 28].

$$\pi = \left[1 - \frac{1}{3} + \frac{1}{5} - - - - - - \pm \frac{1}{2K-1} \mp \frac{K}{4K^2+1}\right]$$

Alternatively, above series for $K = (n+1)/2$ or $n = 2K-1$ may be expressed as

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - - - - - - \pm \frac{1}{n} \mp \frac{(n+1)/2}{(n+1)^2+1}\right]$$

Last but one term is negative or positive according as K is even or odd.

If we take $n = 1$, Pie comes out to be 3. 141564 which is correct to four decimal places,

Other rational approximation given by *Mādhava is*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - - - - - - \pm \frac{1}{n} \mp \frac{\left(\frac{n+1}{2}\right)^2 + 1}{\left[4\left(\frac{n+1}{2}\right) + 1\right]^2 \left[\frac{n+1}{2}\right]}$$

He also gave infinite series

$$\pi = \sqrt{12}\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^2} - - - - - - - - - \infty\right]$$

From the above series *Mādhava* gave a rational approximation to Pie as [28]

$$\pi = \frac{2827433388253}{9^{11}} = 3.14159265359$$

Which is correct to eleven decimal places.

Nīlkaṇtha Somayāji (b. 1444 A.D.) in his *Tantra Sangraha* gave some series expansions for Pie [28, 35, 37]:

$$\frac{\pi}{8}=\frac{1}{2^2-1}+\frac{1}{6^2-1}+\frac{1}{10^2-1}------$$

$$=\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}------$$

$$\frac{4-\pi}{8}=\frac{1}{4^2-1}+\frac{1}{8^2-1}+\frac{1}{12^2-1}+------$$

$$\frac{\pi-3}{4}=\frac{1}{2.3.4}-\frac{1}{4.5.6}+\frac{1}{6.7.8}-\frac{1}{8.9.10}------$$

$$\frac{\pi-3}{6}=\frac{1}{(2.2^2-1)^2-2^2}+\frac{1}{(2.4^2-1)-4^2}+\frac{1}{(2.6^2-1)^2-6^2}+------$$

$$\frac{\pi}{16}=\frac{1}{1^5+4.1}-\frac{1}{3^5+4.3}+\frac{1}{5^5+4.5}-----$$

German mathmetician *Vieta* (b. 1540 A.D.) showed that Pie can be found from the following formula[20]:

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}\frac{1}{2}}}}-----$$

English mathematician *John Wallis* (1616-1703 A.D.) also showed a number pattern, which was entirely new way of closing in on the value of Pie [20]

π 2.2.4.4.6.6.8.8-------
2 1.3.35.5.7.7.9.9·

Lord Brouncher searched for a formula for Pie [20].

$$\pi = \cfrac{4}{1+\cfrac{1}{2+\cfrac{4}{\cfrac{25}{2+\cfrac{49}{\cfrac{81}{2+\cfrac{81}{121+etc.}}}}}}}$$

Saṅkara Verma (c. 1819A.D.) in *Sadrathnmālā* [28] gave Pie as

$$\frac{314159263358979324}{10^{17}} = 3.14159265558979324$$

which is correct to seventeen decimal places.

Lindemann (b.1852 A.D. in 1882 is credited for proving the transcendence of Pie [13, 20, 24].

Svāmī Bhārti Krsṇa Tirtha (1884-1960 A.D.) the *Sankarācārya of Govardhan Math,* gave Pie in *Katapayādi* notation [2, 28]. To know more about *Katapayādi* system [27, pp. 175-176]

गोपी भाग्य मधुव्रात शृङ्गिशो दधिसन्धिग ।
खलजीवित खाताव गलहालारसन्धर ।। ([2],) p. 320)

This gives the Pie for thirty one decimal places, that is,

$\pi = 3.14159265358979323846264338327 92\cdot$

Ramanujan (b, 1887) also obtained some approximations for Pie [28-31]

Some finite expansions for Pie are:

$$\frac{19}{16}\sqrt{7} = 3.14180.$$

$$\frac{99}{88}\left(\frac{7}{7-3\sqrt{2}}\right) = 3.14159274.$$

$$\frac{355}{113}\left(1-\frac{0003}{5533}\right)=3.1415926535897943.$$

He also gave infinite expansions for $\frac{1}{\pi}$ [28, 30]

$$\frac{16}{\pi}=5+\frac{47}{64}\left(\frac{1}{2}\right)^3+\frac{89}{64}\left(\frac{1.3}{2.4}\right)^3+-----------------(1)$$

$$\frac{1}{2\pi\sqrt{2}}=\frac{1103}{99^2}+\frac{27493}{99^6}\frac{1}{2}\frac{1.3}{3^2}+\frac{53883}{99^{10}}\frac{1.3}{2.4}\frac{1.3.57}{4^2 8^2}+-----------(2)$$

In series (2),

$$\frac{1103}{99^2}=0.\ 11253953951,$$

while $\frac{1}{2\pi\sqrt{2}}=0.11253953951,$

which is correct to eight decimal places even by taking only I term.

This very series (2) was used in 1986 to compute Pie for 17 million places (*Ganita Bhārati 1994, 84-85).*

In 1998, *Jasumasa Kanad* (Japanese) found the Pie for 20,1326000 decimal places [24].

In 1989 *David Chudanovaski and* Grary Gregory of Clombia University found the pie for 1011196691 decimal places [38]. Surprisingly, in 1995 *Takahashi* and *Kanada* obtained 3221225466, 4294967286 and 6442450938 digits for Pie which were extended by them in 1997 to 51539600000 digits.

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MAGNETIC BEHAVIOUR OF TERBIUM-IRON-TRANSITION METAL MIXED OXIDES

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ABSTRACT

The compounds of Terbium-iron-transition metal mixed oxides of the type $TbFeTO_4$ have been prepared by solid state reaction technique and characterised by *XRD* pattern. The molar magnetic susceptibility (χ_m) of the powdered sample have been reported in the temperature range 300K to 1100. All studied materials show a typical ferrimagnetic behaviour and χ_m^{-1} vs *T* plot can be expressed in terms of standard relations for ferrimagnetic materials. The slope of asymptotic line to the curve, yield an average magneton number which indicates that all materials are perfectly ionic. The transition metal ions are Fe^{2+} and Fe^{3+} in $TbFe_2O_4$, Fe^{2+} and Cr^{3+} in $TbFeCrO_4$, Fe^{2+} and Mn^{3+} in $TbFeMnO_4$, and Fe^{3+} and Co^{2+} in $TbFeCoO_4$,.

Keywords: Magnetic susceptibility, Terbium-iron-transition metal mixed oxides.

Physics Subject Classification: 07.55 - W. 75.30 Cr

INTRODUCTION

Rare-earh and transition elements are characterised by their unfilled 4f and 3*d* orbitals respectively. Due to this reason, they yield variety of materials with interesting and useful magnetic properties which have not only enriched our understanding but have found numerous application [1-3]. We have prepared a series of rare-earth transition metal compounds with general formula $R_ETT'O_4$ where R_E stands for rare earth, *T* and *T'* for transition metals. We have already reported the electircal transport studies of $YFe\ TO_4$ and $GdFeTO_4$ where *T=Fe, Cr, Mn* and *Co* [4-5] and magnetic behaviour of $GdFeTO_4$ and $YFeTO_4$ [6-7]. This paper reports results of our study on the magnetic behaviour of compounds with general formula $TbFeTO_4$ with T=*Fe, Cr, Mn* and *Co*. So far, no study of this kind has been reported on these compounds.

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MATERIAL PREPARATION AND CHARACTERIZATION

The starting materials for the preparation of these compounds were Tb_2O_3, Fe_2O_3, MnO_2 and CoO which are 99.9% pure. The stoichiometric amount of these oxides were mixed and heated in a silica crucible for 50-hours at temperature of about 1400K. In this process, the mixture was subjected to one intermediate grinding and the final product was cooled down at slow rate. The prepared compounds under go the following solid state reaction.

$$2Tb_2O_3 + 4Fe_2O_3 \xrightarrow[Air]{1400K} 4TbFe_2O_4 + O_2$$

$$2Tb_2O_3 + 2Fe_2O_3 + 2Cr_2O_3 \xrightarrow[Air]{1400K} 4TbFeCrO_4 + O_2$$

$$Tb_2O_3 + Fe_2O_3 + 2MnO_2 \xrightarrow[Air]{1400K} 2TbFeMnO_4 + O_2$$

$$Tb_2O_3 + Fe_2O_3 + 2CoO \xrightarrow[Air]{1400K} 2TbFeCoO_4$$

The weight loss corresponding to loss of oxygen of the right hand side of the above reaction was observed in all cases execpt in $TbFeMnO_4$.In this case, the observed loss was slightly less than expected. The details are described else where[8].

To confirm the complete formation of the prepared compounds *XRD* pattern were obtained for each material using $CuK\alpha$ radiation with $\lambda = 0.15418\ nm$. From *XRD* pattern the values of interplaner spacing d_{hkl} have been obtained using relation.

$$d_{hkl} = \frac{0.15418}{2\sin\theta}(nm) \qquad (1)$$

From d_{hkl} values, structure of the prepared compounds have been resolved using standard procedure[8]. All the peaks have been identified and assigned proper d_{hkl} values. This confirms that prepared compounds are in single phase. All the compounds have been found to have orthorhombic unit cell with unit cell parameters a_0, b_0 and c_0 as given in table1.

Table 1. Structural parameters of the studied $TbFeTO_4$ compounds

Compounds	Lattice Parameters (nm)			Unit Cell	Calculated density
	a_0	b_0	c_0		$Kg\ m^{-3} \times 10^{-3}$
$TbFe_2O_4$	0.6246	0.7366	0.8836	Orthorhombic	5.34
$TbFeCrO_4$	0.6340	0.7482	0.8850	Orthorhombic	5.14
$TbFeMnO_4$	0.6174	0.7286	0.8850	Orthorhombic	5.05
$TbFeCoO_4$	0.6280	0.7410	0.8915	Orthorhombic	5.20

MEASUREMENT OF MAGNETIC SUSCEPTIBILITY

Magnetic susceptibility measurement has been done using Faraday method. In this method one requires only 50-100 mg of the sample as well as the standard materials. The set up consists of a magnetic balance (of 10^{-5} gm accuracy with all control external, from Keroy, India), a tapered pole pieces (2" diameter from 4" base at angle of 30°) electromagnet and constant current power supply (both from Pollytronic, India). The pole gap is variable but has been fixed at a separation of 4.8 cm. At this pole gap, the magnetic field in constant $H\,(dH/dz)$ region, which lies about 2.5 cm above the center of the pole is about 1.5×10^5 Am^{-1} at 2A current in the electromagnet coil. A pyrex tube of 2 cm long and 0.5 cm dia has been used as a sample holder. $Gd_2(WO_4)_3$ with molar magnetic susceptibility value 6.85×10^{-7} at 300 K has been used as standard substance.

The standard material and the sample is hanged from the hook provided in the pan of the balance in constant $H\,(dH/dz)$ region in the pyrex tube and weight is measured in both cases, with and without applied magnetic field. The molar magnetic susceptibility of the sample have been obtained from the relation [8, 9].

$$\chi_m = (\Delta m / \Delta m_S)(m_S / m)\chi_S$$

Δm and Δm_s are changes in the weights of the sample and standard substance, m and m_s are their masses and χ_s the molar magnetic susceptibility of the standard substance. Being a relative method most of the error are automatically eliminated except the error in the measurement of mass m and weight change Δm. The maximum probable error in these measurements has been about 2% at lower temperature (T<500 K). But it increases with the increase of temperature, because hot air movement disturbs the sample holder in spite of closed one end of the furnace, and becomes as high as 5% around 1000K [8,10].

RESULTS AND DISCUSSION

The molar magnetic susceptibility (χ_m) of all the componds have been measured in both heating and cooling cycles. No hysteresis was observed in χ_m and values were found to be almost same in both heating and cooling cycles. However a small weight loss was noticed in heating cycle probably due to presence of moisture. The results are presented in figure 1 as χ_m^{-1} vs T plot. It is seen from this figure that nature of all these plots are essentially similar. In general, χ_m^{-1} *vs* T plots are linear at higher side of temperature. However, there is systematic trends of experimental points towards temperature axis at lower side of temperature. The curves are thus similar to a standard ferrimagnetic material and systematic downward trend is due to the onset of short range magnetic interaction at lower side of temperature. We have tried to fit the experimental points to the standard equation of ferrimagnetism [11] given below by choosing the suitable parameters.

$$\frac{1}{\chi_m}=\frac{T-\theta_{2a}}{\bar{C}_M}-\frac{\theta_b^2}{\bar{C}_M(T-\theta)} \tag{2}$$

where $\bar{C}_M$ is the average value of Curie constant, θ_a is the asymptotic Curie temperature and θ_b and θ are parametric temperature. The curves using eq. 2 are drawn by full line in respective χ_m^{-1} vs T plots. It is seen that experimental point can be well fitted by eq. 2 over wide temperature range. The values of $\bar{C}_M$, θ_a,θ_b and θ are given in table 2. The ferrimagnetic Curie (or Neel) temperature have been evaluated using condition $T \to T_c$, $\chi_m^{-1} \to 0$. This gives.

$$(T_c - \theta_a)(T_c - \theta) = \theta_b^2 \tag{3}$$

The real and positive value of T_c are meaningful and have been evaluated using above relation. These values are also given in table 2.

Table 2. Magnetic parameters of studied compounds

Compounds	θ_a (K)	θ (K)	θ_b (K)	T_c (K)	$\overline{C_M}$ $(mol^{-1}m^3k)x10^4$
$TbFe_2O_4$	-348	236	85	252	2.333
$TbFeCrO_4$	-937	243	116	255	2.050
$TbFeMnO_4$	-990	165	194	198	2.167
$TbFeCoO_4$	-540	245	134	267	2.250

The compounds $TbFeTO_4$ contains only three types of magnetic ions Tb^{3+}, Fe^{3+} or Fe^{2+} and T^{3+} or T^{2+}. Thus at temperature much higher to T_c, the molar magnetic susceptibility of these compounds can be expressed by the relation.

$$\chi_m = \frac{N\mu_\beta^2\mu_0}{3_k}\left[\frac{P_1^2}{T-\theta_{a1}} + \frac{P_2^2}{T-\theta_{a2}} + \frac{P_3^2}{T-\theta_{a3}}\right] \tag{4}$$

where N is the Avogadro number, μ_β is Bohr magneton, μ_0 is the permeability constant, k is Boltzman constant, P_1, P_2 and P_3 are the magneton numbers of three types of magnetic ions respetively and $\theta_{a1,}\theta_{a2}$ and $\theta_{a3,}$ are paramagnetic Curie temperatures which takes into account the effect of various interaction. Assuming $\theta_{a1,} = \theta_{a2} = \theta_{a3}$ we can write above equation as.

$$\chi_m^{-1} = \frac{k(T-\theta_a)}{N\mu_\beta^2\,\mu_0\overline{P}^2} \tag{5}$$

where $\overline{P}^2 = \dfrac{P_1^2 + P_2^2 + P_3^2}{3}$ is the average effective magneton per ion.

Comparing this equation with asymptotic equation of the curve given by eq.2, one gets.

$$\theta = \theta_a \text{ and } \overline{C}_M = N\mu_\beta^2\mu_0\overline{P}^2 / k$$

$$\text{or} \quad \overline{P} = \left[k\overline{C}_M / N\mu_2^\beta \, \mu 0\right]^{1/2} \qquad (6)$$

The experimental value of $\overline{P}$ can be calculated from the evaluated value of $\overline{C}_M$. The theoretical values of P_1, P_2 and P_3 are known. Hence, one can obtain the theoretical values of $\overline{P}$. The experimental and theoretical values of $\overline{P}$ are given in table 3 together with the magnetic ions used to obtain theoretical values of $\overline{P}$.

It is seen from this table that there is a good agreement between the theoretical and experimental values of $\overline{P}$. This indicates that all the studied compounds are essentially ionic and magnetic states of the ions are as indicated in the fourth column of the table 3. It can also be noticed that in $TbFeCrO_4$ and $TbFeMnO_4$ compounds, there exist Cr^{3+} and Mn^{3+} ions and they substitute Fe^{3+} ions. This is quite reasonable in view of the natural valency of these elements.

Table 3. Magnetic ions and average effective magneton number per ion ($\overline{P}$)

Compounds	Values of $\overline{P}$		Magnetic ions
	Theo.	Expt.	
$TbFe_2O_4$	7.14	7.05	$Tb^{3+}Fe^{3+}Fe^{2+}$
$TbFeCrO_4$	6.66	6.61	$Tb^{3+}C_r^{3+}Fe^{2+}$
$TbFeMnO_4$	6.88	6.79	$Tb^{3+}Mn^{3+}Fe^{2+}$
$TbFeCoO_4$	6.93	6.92	$Tb^{3+}Fe^{3+}Co^{2+}$

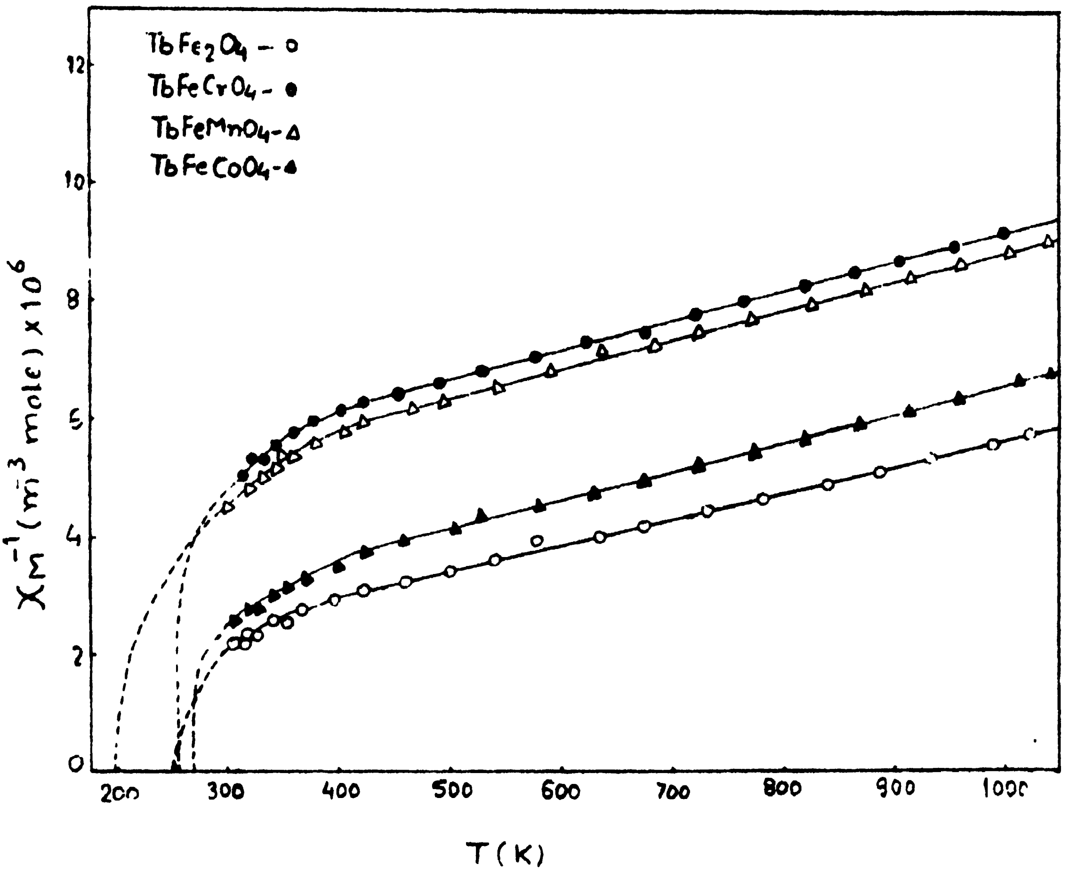

Fig. 1. Plots of inverse of magnetic susceptibility against absolute temperature for TbFe$_2$ O$_4$, TbFeCrO$_4$, TbFeMnO, and TbFeCoO,

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COINCIDENCES AND FIXED POINTS OF MEIR-KEELER TYPE CONTRACTIVE MAPPINGS IN RANDOM NORMED SPACES

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ABSTRACT

In this paper we introduce the notion of Meir-Keeler type contractive condition for four mappings in a random normed space and obtain a coincidence and fixed point theorem for these mappings. Our result improves and extends some of the known results in metric, Menger and random normed spaces.

Key words and Phrases: Random normed space, Menger space, fixed point.

Mathematics Subject Classifications (2000): 54H25, 54E70, 47H10, 46S50.

INTRODUCTION

Serstnev [11] introduced the concept of a random normed space (*RN*-space) as a probabilistic generalization of the concept of normed linear spaces. The study of fixed point theorems in *RN*-spaces was initiated by Bocșan [1] and further results have been established in [2, 3]. On the other hand, Meir and Keeler [6] presented the following remarkable generalization of the celebrated Banach contraction principle:

Let f be a self mapping of a metric space (M, d). If for an $\varepsilon > 0$, there exists a $\delta > 0$ such that $\varepsilon \le d(u,v) < \varepsilon + \delta$ implies $d(fu, fv) < \varepsilon$ for every u, v in M, then f has a fixed point.

This result has been studied extensively and extended by several authors (see, for instance, [5], [7-10]). An excellent bibliography is given by Jachymski [4].

The notion of Meir-Keeler type contractive condition for a triplet of mappings on a Menger space was introduced in [13] and some coincidence and fixed point theorems were proved. In this paper we introduce the notion of Meir-Keeler type contractive condition for a quadruplet of mappings on an arbitrary set with values in

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an RN-space and prove a coincidence theorem for such mappings. We further, derive the existence of common fixed point of these mappings commuting only at their coincidences. These results, being new in this direction, are interesting generalizations and improvement of a number of known results in different settings.

PRELIMINARIES

Let $\mathcal{L}$ denote the set of all distribution functions. An RN-space is a triplet $(S, \mathcal{F}, t)$ consisting of a real or complex linear space S, the mapping $\mathcal{F}: S \rightarrow \mathcal{L}$ and the t- norm t satisfying the following conditions in which $\mathcal{F}(u)$ is denoted by F_u, and $F_u(x)$ is interpreted as the probability of the norm of u being less than x:

(a) $F_u(0) = 0$ for all $u \in S$;

(b) $F_u(x) = H(x)$ iff $u = 0$, where $H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0; \end{cases}$

(c) $F_{qu}(x) = F_u(x/|q|)$, where q is a non-zero scalar and $x \in R$;

(d) $F_{u+v}(x+y) \geq t\{F_u(x), F_v(y)\}$ for all u, v in S and $x, y \geq 0$;

(e) $t(x, y) \geq \max\ \{x+y-1, 0\}$ for all $x, y \in [0, 1]$.

For detailed topological preliminaries of RN-spaces, we refer to [11].

Suppose the mappings f, g, h and k on an arbitrary set with values in an RN-space satisfy the following condition :

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$G_{u-v}(\varepsilon) = 0$ and $G_{u-v}(\varepsilon+\delta) > 1-\lambda$ imply $F_{fu-gv}(\varepsilon) > 1-\lambda$, where

$$G_{u-v}(\varepsilon) = \min\ \{F_{hu-kv}(\varepsilon), F_{fu-hu}(\varepsilon), F_{gv-kv}(\varepsilon), F_{fu-kv}(2\varepsilon), F_{gv-hu}(2\varepsilon)\}. \quad (1)$$

The condition introduced above will be referred to as Meir-Keeler contractive condition for a quadruplet of mappings and it is evidently equivalent to

$$F_{fu-gv}(\varepsilon) > \min\{F_{hu-kv}(\varepsilon), F_{fu-hu}(\varepsilon), F_{gv-kv}(\varepsilon),\ F_{fu-kv}(2\varepsilon), F_{gv-hu}(2\varepsilon)\}. \tag{2}$$

Throughout this paper X stands for an arbitrary nonempty set and S for an RN-space. By $C(f, h)$ we denote the set of coincidence points of f and h, i.e. $C(f, h) = \{r \in x : fr = hr\}$.

The following result for $f, g, h, k : X \to S$ satisfying the most general contractive condition of Meir-Keeler type ((1) or (2)) improves, among others, the results of [3] and [12] on one hand and generalizes the results of [13] on the other.

Theorem. Let $f, g, h, k : X \to S$ be a Meir- Keeler type contractive quadruplet such that

$$f(X) \cup g(X) \subset h(X) \cap k(X); \tag{3}$$

$$h(X) \cap K(X) \text{ is a complete subspace of } S. \tag{4}$$

Then

(i) f and h have a coincidence point ;

(ii) g and h have a coincidence point.

Further, if $X=S, fhu = hfu$, $u \in C(f, h)$ and $gkv = kgv$, $v \in C(g, k)$ then

(iii) f, g, h and k have a unique common fixed point.

Proof. For $u_0 \in X$, we define the sequences $\{u_n\}$ and $\{v_n\}$ as follows:

$$fu_{2n} = ku_{2n+1} = v_{2n},\ gu_{2n+1} = hu_{2n+2} = v_{2n+1},\ n = 0,\ 1, 2, \ldots .$$

This can be done by virtue of (3). Now from (2),

$$\begin{aligned} F_{v_{2n}-v_{2n+1}}(\varepsilon) &= F_{fu_{2n}-gu_{2n+1}}(\varepsilon) \\ &> \min\{F_{v_{2n-1}-v_{2n}}(\varepsilon),\ F_{v_{2n}-v_{2n-1}}(\varepsilon),\ F_{v_{2n+1}-v_{2n}}(\varepsilon),\ F_{v_{2n}-v_{2n}}(2\varepsilon),\ F_{v_{2n+1}-v_{2n-1}}(2\varepsilon)\} \\ &= F_{v_{2n-1}-v_{2n}}(\varepsilon), \end{aligned}$$

as $\quad F_{v_{2n-1}-v_{2n+1}}(2\varepsilon) \geq \min\{F_{v_{2n-1}-v_{2n}}(\varepsilon), F_{v_{2n}-v_{2n+1}}(\varepsilon)\}$.

Similarly

$$F_{v_{2n+1}-v_{2n+2}}(\varepsilon) > F_{v_{2n}-v_{2n+1}}(\varepsilon).$$

Thus $F_{v_n-v_{n+1}}(\varepsilon)$ is a monotone increasing sequence. Since F is a distribution function, $\sup F_{v_n-v_{n+1}}(\varepsilon) = 1$. Therefore $F_{v_n-v_{n+1}}(\varepsilon) \to 1$.

To prove that $\{v_n\}$ is a Cauchy sequence, suppose the contrary to be true. Then there exists $\varepsilon'\ (= 2\varepsilon)$ such that for each integer N and integers $m, n \geq N$ we have

$$F_{v_m-v_n}(2\varepsilon) > 1 - \lambda,\ 0 < \lambda < 1. \tag{5}$$

We choose $\delta,\ 0 < \delta < \varepsilon$, such that (1) is satisfied. Since

$F_{v_n - v_{n+1}}(\delta / 6) \to 1$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that for $m, n \geq N$

$$F_{v_m - v_{m+1}}(\delta / 6) > 1 - \lambda \text{ and} F_{v_m - v_n}(\varepsilon + \delta) < 1 - \lambda . \tag{6}$$

Let (6) be true for an even n. Then $F_{v_m - v_{n+1}}(\varepsilon + \delta / 3) < 1 - \lambda$, by (5).

Thus we have a smallest odd integer $j > n$ such that

$$F_{v_n - v_j}\left(\varepsilon + \delta / 3\right) < 1 - \lambda \text{ and } F_{v_n - v_{j-2}}\left(\varepsilon + \delta / 3\right) > 1 - \lambda. \tag{7}$$

Therefore from (6) and (7),

$$F_{v_n - v_j}(\varepsilon + 2\delta / 3) \geq \min\left\{F_{v_n - v_{j-2}}(\varepsilon + \delta / 3), F_{v_{j-2} - v_{j-1}}(\delta / 6), F_{v_{j-1} - v_j}\left(\delta / 6\right)\right\}$$

$$> \min\{1 - \lambda,\ 1 - \lambda,\ 1 - \lambda\} = 1 - \lambda .$$

Now $G_{v_{j+1} - v_{n+1}}(\varepsilon + \delta) = G_{fu_{j+1} - gu_{n+1}}(\varepsilon + \delta)$

$$= \min\ \{ F_{v_j - v_n}(\varepsilon + \delta), F_{v_{j+1} - v_j}(\varepsilon + \delta), F_{v_{n+1} - v_n}(\varepsilon + \delta),$$

$$F_{v_{j+1} - v_n}(2(\varepsilon + \delta)), F_{v_{n+1} - v_j}(2(\varepsilon + \delta))\}$$

$$> \min\{F_{v_j - v_n}(\varepsilon + 2\delta / 3), F_{v_{j+1} - v_j}(\delta / 6), F_{v_{n+1} - v_n}(\delta / 6),$$

$$F_{v_{j+1} - v_j}(\varepsilon + \delta), F_{v_j - v_n}(\varepsilon + \delta), F_{v_{n+1} - v_n}(\varepsilon + \delta), F_{v_n - v_j}(\varepsilon + \delta)\} > 1 \text{-} \lambda .$$

Therefore from (1),

$$F_{fv_{j+1} - gv_{n+1}}(\varepsilon) = F_{v_{j+1} - v_{n+1}}(\varepsilon) = F_{v_{n+1} - v_{j+1}}\left(\varepsilon\right) > 1 - \lambda.$$

So

$$F_{v_n - v_j}(\varepsilon + \delta / 3) \geq \min\{F_{v_n - v_{n+1}}(\delta / 6), F_{v_{n+1} - v_{j+1}}(\varepsilon), F_{v_{j+1} - v_j}(\delta / 6)\} > 1 - \lambda,$$

contradicting (7).

Therefore $\{v_n\}$ is a cauchy sequence in $h(X) \cap k(X)$. Let z be the limit. Then there exist point r, s such that $r = h^{-1}z$ and $s = k^{-1}z$, giving $z = hr$.

To prove $fr = z$, let $U_{fr}(\varepsilon, \lambda)$ be any neighbourhood of fr. Since $v_{2n} \to hr$, there exists an integer $N'(\varepsilon, \lambda)$ such that $2n \geq N'$ implies

$$F_{hu_{2n} - hr}(\varepsilon) > 1 - \lambda, \text{ and } F_{ku_{2n+1} - hr}(\varepsilon) > 1 - \lambda. \tag{8}$$

Now by (2),

$$F_{fr - gu_{2n+1}}(\varepsilon) > \min\{F_{hr - ku_{2n+1}}(\varepsilon), F_{fr - hr}(\varepsilon), F_{gu_{2n+1} - ku_{2n+1}}(\varepsilon),$$

$$F_{fr - ku_{2n+1}}(2\varepsilon), F_{gu_{2n+1} - hr}(2\varepsilon)\}$$

$$\geq \min\{F_{hr-ku_{2n+1}}(\varepsilon), F_{fr-hu_{2n+2}}(\varepsilon/2), F_{hu_{2n+2}-hr}(\varepsilon/2), F_{gu_{2n+1}-hr}(\varepsilon/2),$$

$$F_{hr-ku_{2n+1}}(\varepsilon/2), F_{fr-gu_{2n+1}}(\varepsilon), F_{gu_{2n+1}-ku_{2n+1}}(\varepsilon), F_{gu_{2n+1}-hr}(2\varepsilon)\} > 1-\lambda.$$

Thus $hr = z = fr$. Similarly $ks = z = gs$.

To prove (iii), we note that $r \in C(f, h)$, $s \in C(g, k)$ and

(a') $hr = fr = z = ks = gs$,

(b') $fz = fhr = hfr = hz$,

(c') $gz = gks = kgs = kz$.

Now replacing u and v by r and z respectively in (2) and invoking (a')-(c'), we get

$F_{fr-gz}(\varepsilon) > F_{z-gz}(\varepsilon)$, implying, $gz = z$. By (c'),

$kz = gz = z$.

Similarly taking $u = z$, $v = s$, we obtain $fz = z$.

Thus by (b') and (c')

$fz = hz = z = gz = kz$.

The uniqueness of z as the common fixed point of f, g, h and k can be easily seen from (2).

Taking $X = S$, $h = k = I$ and S to be orbitally complete (see,[12]) in the above result, we get the following corollary.

Corollary. Let the self mappings f, g on an (f, g)- orbitally complete RN-space S satisfy

$$F_{fu-gv}(\varepsilon) > \min\{F_{u-v}(\varepsilon), F_{fu-u}(\varepsilon), F_{gv-v}(\varepsilon), F_{fu-v}(2\varepsilon), F_{gv-u}(2\varepsilon)\}.$$

Then for every $u_0 \in S$, either

f or g has a fixed point in the (f, g)- orbit of u_0 or

f and g have a unique common fixed point z such that $\lim u_n = z$.

The above result for a pair of mappings is an extension to RN-spaces of Theorem 2 of [10] under the most general contractive condition of Meir-Keeler type. This result is also an improvement over some of the known results, including that of [10], in the sense that we do not use the continuity of either mapping here.

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FIXED POINTS OF PROBABILISTIC DENSIFYING MAPPINGS

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ABSTRACT

In this paper we prove some fixed point theorems for densifying mappings on probabilistic metric spaces. Several results for densifying mappings in metric and probabilistic metric spaces are obtained as special cases.

Mathematics Subjects Classifications (2000): 54H25, 54E70, 47H10

Key-words and Phrases: Probabilistic densifying mappings, probabilistic weakly ϕ-contractive, fixed point.

INTRODUCTION

The notion of densifying (also called condensing) mappings was introduced by Sadovskii [11]. Furi and Vingoli [4] studied the fundamental properties of Kuratowski measure of noncompactness of a bounded set in a metric space. Later, several mathematicians investigatedfixed point theorems for densifying mappings [6, 8, 9, 10,13, 14].

Bocsan and Constantin [3] extended Kuratowski measure of non-compactness of probabilistic metric spaecs (PM-spaces). The concept of probabilistic densifying mapping was introduced by Bocsan [1] (cf. [2]), and some fixed poing theorems for these mappings have been proved [2, 5, 7, 12, 13].

In this paper we introduce the notion of $(p; q_1, q_2, \ldots\ldots, q_m)$- probabilistic densifying mapping and prove a fixed point theorem for such mappings. Further we extend this result for a pair of mappings. The paper is concluded with a result which is the generalization of a result of Pant, Tivari and Singh [7]. Several fixed point theorems for densifying mappings on metric and PM-spaces may be obtained as special cases of our results.

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DEFINITIONS AND RESULTS

Definition 1 [1]. Let A be a nonempty subset of X, A function $D_A(.)$ defined by

$$D_A(X) = \sup_{t<x} \inf_{p,q \in A} F_{p,q}(t)$$

is called the probabilistic diameter of A.

A is said to be bounded if $\sup_{x<R} D_A(x) = 1$, R being the set of real numbers.

Definition 2 [3]. For a probabilistic bounded subset A of X, the function $\alpha_A(.)$ defined by $\alpha_A(x) = \sup\{\varepsilon \geq 0 | \exists$ a finite cover A of A such that $D_s(X) \geq \varepsilon$ for all $S \in A\}$ is called Kuratowski function.

The following properties of α_A are proved in [10].

(i) $\alpha_A(X) \in \zeta$ for every probabilistic bounded subset A of X and for all $x \in R$;

(ii) $\alpha_A(X) \geq D_A(X)$ for every probabilistic bounded subset A of X and for all $x \in R$;

(iii) If $\phi \neq A \subset B \subset X$ then $\alpha_A(x) \geq \alpha_B(x)$ for all $x \in R$;

(iv) If A and B are probabilistic bounded, then for each $x \in R$

$\alpha_{A \cup B}(X) \geq \min\{\alpha_A(X), \alpha(X)\}$;

(v) If A is a probabilistic bounded subset of X, then

$\alpha_{\bar{A}}(X) \geq \alpha_A(X)$;

(vi) A probabilistic bounded subset A of X is probabilistic precompact iff

$\alpha_{\bar{A}} = H$;

(vii) If the t-norm t is 't-min' then $\alpha_{\bar{A}} = \alpha\, \overline{COA}$ for every probabilistic bounded subset A of X, where $\overline{COA}$ denotes the closed convex hull of A in the (ε, λ) -topology on X.

Definition 3 [1]. Let (X, F) be a probabilistic metric space. A continuous mapping f of

X into X is called a probabilistic densifying mapping iff for every subset A of X such that

$\alpha_A < H$ we have $\alpha_{f(a)} > \alpha_A$

We introduce the following:

Definition 4. A mapping $f: X \rightarrow$ is said to be $(p; q_1 q_2 \ldots\ldots q_m)$ probabilistic densifying if for $A \subset X$,

(1) f^p is continuous and

(2) $\alpha_{f^p(A)} > \sum_{j=1} a_j \alpha_{f_j^p(A)}$

whenever $\sum a_j \alpha_{f^q j(A)} < H$ where $p, q_1, q_2 \ldots q_m$ are all non- negative integers and the a_j' s are non- negative reals such that $\sum_{j=1}^{m} a_j = 1$.

Definition 5. A mapping $f: X \rightarrow X$ is called probabilistic weakly $\varnothing$ – contractive iff

(3) for every $A \subset X, \alpha_A < H$ implies $\alpha_{f(A)} < \alpha_A$

(4) for every $u, v \in X, u \neq v$

$\varnothing (f(u), f(v)) > \varnothing (u, v)$

where $\varnothing$ is a τ–Continuous mapping [3] of XxX into $\mathcal{F}$. It is to be noted that τ - continuity coincides with upper semicontinuity [5].

THEOREM 1. Let f be a $(p; q_1, q_2 \ldots q_m)$ – probabilistic densifying mapping defined on a complete Menger space $(X, \mathcal{F}, t)$ wherein $\sup_{x<1} t(x, x) = 1$ such that f^p is probabilistic weakly $\varnothing$-contractive. If, for some $u_o \in X$, the sequence of iterates $\{u_n\}$ is bounded, then f has a unique fixed point in X.

PROOF. Let $A = \bigcup_{n=0}^{\infty} \{u_n\}$ where $u_n = fu_{n-1}, n = 1, 2, \ldots$, then

$$f^p(A) = f^p u_0, f^{p+1} u_0, \ldots\ldots\ldots, \quad f^{q_j}(A) = f^{q_j} u_0, f^{q_j+1} u_0, \ldots\ldots$$

Thus $f^p(A)$ and $f^q{}_j (A), j = 1, 2, \ldots, m$ all differ from A only by a finite number of terms. Hence

$$\alpha_A = \alpha_{f^p(A)} = \alpha_{f^{q_j}(A)}$$

If $\sum a_j \alpha_{f^{q_j}(A)} < H$, then by (2),

$$\alpha f^{p(A)} > \sum_{j=1}^{m} a_j \alpha_{f^{q_j}(A)} = \alpha_{f^{p}(A)}$$ which is, a contradiction.

Therefore $\overline{f^p(A)}$ is compact.

We consider $\psi : f^p(A) \to \mathcal{L}$ defined by

$$\psi(u) = \varnothing(u, f^p(u)).$$

Then ψ is u.s.c and attains a maximum value at some point, say, $z \in f^p(A)$.

The continuity of f^p gives

$$f^p\,\overline{f^p(A)} \subset f^p(f^p(A)) = \overline{f^{2p}(A)} \subset \overline{f^p(A)},$$ i.e. $f^p(z) \in \overline{f^p(A)}$.

If $Z \neq f^p(z)$, then

$$\psi(f^p(z)) = \varnothing(f_f^p(z), f^{2p}(z)) > \varnothing(z, f^p(z)) = f(z).$$ This contradicts the maximality of ψ at z. Hence $z = f^p(z)$.

Using the probabilistic weak $\varnothing$-contractivity of f^p it is easy to see that z is the unique fixed point of f^p.

Also $f(x) = f(f^p(z))$. Hence $z = f(z)$ by the uniqueness of z.

REMARK 1. If f is a (1; 0) probabilistic densifying mapping, then the above Theorem will reduce to the following:

COROLLARY. Let f be probabilistic densifying and probabilistic weakly $\varnothing$-contractive mapping defined on a complete Menger space $(X, \mathcal{F}, t)$ with $\sup_{x<1} t(x, x) = 1$. If for some $u_0 \in X$, the sequence of iterates $\{u_n\}$ is bounded, then f has a unique fixed point in X.

Now we extend Definition 4 for a pair of mappings.

Definition 6. A pair of mappings $S = \{f, g\}$ defined on a Menger space $(X, \mathcal{F}, t)$ with

$\sup_{x<1} t(x, x) = 1$ is called $(p; q_1, q_2, \ldots\ldots\ldots, q_m)$ - *probabilistic*

densifying if for $A \subset X$,

(5) f^p and g^q are continuous and

$$(6) \quad \alpha_{(f^p g^q)(A)} > \sum_{j=1}^{m} a_j \alpha_{(f^{p_j} g^{q_j})(A)}$$

whenever $\Sigma \alpha_{(f^p g^q)(A)} < H$, where $p, q_1, q_2, \ldots\ldots q_m$ are are all non-negative integers and the a_j's are non-negative reals such that $\sum_{j=1}^{m} a_j = 1$.

THEOREM 2. Let $S = \{f, g\}$ be a pair of commutative and $(p; q_1, q_2 \ldots\ldots q_m)$-probabilistic densifying mapping on a complete Menger space $(X, \mathcal{F}, t)$ with $\sup_{x<1} t(x, x) = 1$ such that $f^p g^q$ be probabilistic weakly $\varnothing$-contractive. Further, let for some $u_0, u_n = fu_{n-1}$ if n is odd and, $u_n = gu_{n-1}$ if n is even, then f and g have unique common fixed point.

PROOF. Let $A = \bigcup_{n=0}^{\infty} \{u_n\}$.

Now $f^p g^q$ (A) and $(fg)^{q_j}(A), j = 1, 2, \ldots\ldots, n$, all differ from A only finite number of terms. Hence,

$$\alpha_A = \alpha_{(f^p g^q)(A)} = \alpha_{(fg)^q (A)}$$

If $\quad \sum_{j=1}^{m} a_j \alpha_{(fg)^{q_j}(A)} H$, then by (6)

$$\alpha_A = \alpha_{(f^p g^q)(A)} > \sum_{j=1}^{m} a_j \alpha_{(fg)^{q_j}(A)} = \alpha_A \sum_{j=1}^{m} \alpha_j = \alpha_A \text{, which is absurd.}$$

Therefore $\overline{A}$ is compact.

Define $\psi : \overline{A} \to R$ such that $\psi(u) = \varnothing(u, (f^p g^p)(u))$. Then ψ is u.s.c. on $\overline{A}$ as $\varnothing$ is u.s.c. So ψ has a maximal value at $z \in \overline{A}$.

Now $f^p g^q(\overline{A}) \subset \overline{f^p(g^q(A))} \subset \overline{A}.$. Hence $f^p g^q(z) \in \overline{A}$

If $z \neq f^p g^q(z) \in \overline{A}$, then $\psi\,(f^p g^q(z)) = \varnothing(f^p g^q f(z), f^p g^q f^p g^q(z))$

$> \varnothing(z, f^p g^q(z)) = \psi(z)$, contradiction to the maximality of f at z. Hence $f^p g^q(z)) = z$.

Now we prove the uniqueness of z. Let w be another fixed point of $f^p g^q$, then

$\varnothing(z, w) = \varnothing(f^p g^q z, f^p g^q w) > \varnothing(z, w)$ which is not possible.

Therefore $z = w$.

Finally, we show that z is the unique common fixed point of f and g as well.

Notice that $z = f^p g^q(z)$ implies $f(z) = f(f^p g^q(z)) = f^p g^q(f(z))$.

Thus f (z) is the fixed point of $f^p g^q$. But z has been shown the unique fixed point of $f^p g^q$ and so $z = f(z)$.

Similarly $z = g(z)$. Hence z is the unique common fixed point of f and g.

Definition 7. Let $S = \{f^p, g^q\}$, for some non-negative integer p and q, be a pair of self-mappings on Menger space $(X, \mathcal{F}, t)$ with $\sup_{x<1} t(x, x) = 1$. Fore $u_0 \in X$, the sequence $J_s(u_0) = \{u_0, f^p u_0, g^q f^p u_0, f^p g^q f^p u_0, \ldots\ldots\}$ is called the joint sequence of iterates of S at u_0.

The following lemma is obvious (see also [7, Lemma, p. 152]).

LEMMA. Let $(X, \mathcal{F}, t)$ be a complete Menger space with $\sup_{x<1} t(x, x) = 1$ and f, g be self mappings of X. Then f and g are probabilistic densifying mappings iff for every pair of probabilistic bounded subset A and B of X,

$\alpha_f(A) \cup g(B) > \alpha_{A \cup B}$, whenever $\alpha_{A \cup B} > H$

The following result is an extension of Pant, Tivari and Singh [7, Theorem, p. 152].

THEOREM 3. Let $S = \{f^p, g^q\}$ be a pair of commutative probabilistic densifying mappings defined on a complete Menger space $(X, \mathcal{F}, t)$ such that $\sup_{x>1} t(x, x) = 1$, where p and q are some positive integers. Furhter, let $f^p g^q$ be probabilistic weakly $\varnothing$

contractive and for some $u_0 \in X$, the joint sequence of iterates $J_s(u_0)$ of S at u_0 is bounded, then f and g have a unique common fixed point in X.

PROOF. Let A = $\{u_0, f^p u_0, g^q f^p u_0, \ldots\ldots\ldots\}$,

B = $\{u_0, g^q f^p u_0, \ldots..\}$ and C = $\{f^p u_0, f^p g^q f^p u_0 \ldots..\}$,

so that $A = B \cup C$ and $A = f^p(B) \cup g^q(C) \cup \{u_0\}$.

Now replacing f and g by f^p and g^q respectively in the proof of [7, Theorem, p. 152], we get $f^p g^q(z) = z$. Uniqeness of z can be seen easily.

Rest of the proof follows from Theorem 2.

REMARK 2. By taking = 1, $q = 1$ in the above theorem we get the result in [7, Theorem, p. 152] as special case.

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A GENERALIZATION OF MEIR-KEELER TYPE COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS

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ABSTRACT

In this paper, we introduce a common fixed point principle which includes several known contractive definitions as particular cases and employs a Lipschitz type analogue of known contractive definitions.

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Key words and phrases: Fixed point, compatible mappings.

INTRODUCTION

The study of common fixed point of mappings satisfying contractive type conditions has attracted a great deal of research activity during the last two decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form

$$d(Ax, By) \leq hm(x, y), 0 \leq h < 1, \tag{1}$$

where $m(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), [d(Sx,By) + d(Ax,Ty)]/2\}$,

or, a Meir-Keeler type (ε, δ)-contractive condition of the form

given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varepsilon \leq m(x,y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon$, (2)

or, a ϕ -contractive condition of the form $d(Ax, By) \leq \phi(m(x,y))$, (3)

involving a contractive gauge function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.

Clearly, condition (1) is a special case of both conditions (2) and (3). A ϕ -contractive condition (3) does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, to ensure the existence of common

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fixed point under the contractive condition (3), the following conditions on the function ϕ have been introduced and used by various authors.

(I) $\phi(t)$ is non decreasing and $t/(t-\phi(t))$ is non increasing [2],

(II) $\phi(t)$ is non decreasing and $\lim_n \phi^n/(t)=0$ for each $t > 0$ [4, 8],

(III) ϕ is upper semi continuous [1, 4, 7, 11, 12] or equivalently,

(IV) ϕ is non decreasing and continuous from right ([17]).

It is now known (e.g, [4, 15]) that if any of the conditions (I), (II), (III) or (IV) is assumed on ϕ, then a ϕ-contactive conditions (3) implies an analogous (ε,δ)-contractive condition (2) and both the contractive conditions hold simultaneously. Similarly, a Meir-Keeler type (ε,δ)-contractive condition does not ensure the existence of a fixed point. The following example illustrates that an (ε,δ)-contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous ϕ-contractive condition (3).

EXAMPLE 1. ([15]) Let $X = [0, 2]$ and d be the Euclidean metric co X. Define $f: X \to X$ by $fx = (1+x)/2$ if $x < 1$; $fx = 0$ if $x \geq 1$. Then, it satisfies contractive condition $\varepsilon \geq \max\{d(x,y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\} < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon$, with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon > 1$ but f does not have a fixed point. Also does not satisfy the contractive condition.

$$d(fx, fy) \leq \phi(\max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy)+d(y, fx)]\ 2\}).$$

since the desired function $\phi(t)$ cannot be defined at $t = 1$.

Hence, the two type of contractive conditions (2) and (3) are independent of each other. Thus, to ensure the existence of common fixed point under the contractive condition (2), the following condition on the function δ have been introduced and used by various authors.

(V) δ is non decreasing [10, 11]

(VI) δ is lower semi continuous [5, 6].

Jachymski [4] has shown that the (ε, δ)-contractive condition (2) with a non decreasing δ implies a ϕ-contractive condition (3). Also, Pant *et al.* [15] have shown that the (ε, δ)-contractive condition (2) with a lower semi continuous δ, implies a ϕ-contractive condition (3). Thus, we see that if additional conditions are assumed on δ then the (ε, δ) contractive condition (2) implies an analogous ϕ-contractive condition (3) and both the contractive conditions hold simultaneously.

It is thus clear that contractive conditions (2) and (3) hold simultaneously whenever (2) or (3) is assumed with additional condition on δ or ϕ respectively. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (2) or (3) with additional conditions on δ and ϕ, we assume contractive condition (2) together with a Lipschitz type analogue of condition (3); that is, a condition of the form

$$d(Ax, By) < \max\left\{k_1 d(Sx, Ty), k_2\left[d(Ax, Sx) + d(By, Ty)\right]/2, \left[d(Sx, By) + d(Ax, Ty)\right]/2\right\}$$

$k_1 > 0$ and $1 \le k_2 < 2$. We prove a common fixed point theorem for four mappings using this approach. It is independent of known earlier results. It may be noted that all the known results have been dealt with the case $k_1 = 1$; $k_2 = 1$ and so all such results are obtained as special Case of our theorem.

Two self-mappings A and S of a metric space (X, d) are called compatible (see Jungck [5]) if $\lim_n d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. It is easy to see that compatible maps commute at their coincidence points.

RESULTS

We prove the following theorem with the notation M(x,y) defined as

$$M(x, y) = \max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \left[d(Sx, By) + d(Ax, Ty)\right]/2\right\}$$

Theorem 1. Let (A, S) and (B, T) be compatible pairs of self mappings of a complete metric space (X, d) such that

(i) $AX \subset TX, BX \subset SX$,

(ii) given $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \le M(x,y) < \varepsilon + \delta \Rightarrow d(Ax,By) < \varepsilon$, and

(iii) $d(Ax,By) < \max\{k_1 d(Sx,Ty), k_2[d(Ax,Sx)+d(By,Ty)]/2$

$$[d(Sx,By)+d(Ax,Ty)]/2\}, \text{ for } k_1 > 0, 1 \le k_2 < 2.$$

If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule $y_{2n} = Ax_{2n} = T_{x_{2n+1}}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2.}$ (4)

This can be done by virtue of (i). Then following the argument by Rhoades *et al.* [19] it follows that $\{y_n\}$ is a Cauchy sequence in X. But X is complete so there exists a point z in X such that $y_n \to z$. Also, using (4), we have

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z. \tag{5}$$

Suppose that S is continuous. Then $SSx_{2n} \to Sz, SAx_{2n} \to Sz$ and compatibility of A and S implies that $ASx_{2n} \to Sz$. Also, since $AX \subset TX$, corresponding to each value of n, there exists, z_{2n} in X such that $ASx_{2n} = Tz_{2n}$. Thus $ASx_{2n} = Tz_{2n} \to Sz$ and $SSx_{2n} \to Sz$. We show that $\lim_n Bz_{2n} = Sz$. If not, then there exists a subsequence $\{Bz_{2m}\}$ of $\{Bz_{2n}\}$ a number $r > 0$ and a positive integer N such that for each $m \ge N$, we have $d(ASx_{2m}, Bz_{2m}) \ge r$, $d(Sz, Bz_{2m}) \ge r$ and in view of (iii), we get

$$d(ASx_{2m}, Bz_{2m}) > \max\{k_1 d(SSx_{2m}, Tz_{2m}), k_2[d(ASx_{2m}, SSx_{2m}) + d(Bz_{2m}, Tz_{2m})]/2,$$

$$[d(SSx_{2m}, Bz_{2m}) + d(ASx_{2m}, Tz_{2m})]/2\},$$

which, on letting $m \to \infty$, yields

$$d(Sz, Bz_{2m}) \le [d(Sz, Bz_{2m})]/2 < d(Sz, Bz_{2m}), \text{ a contradiction.}$$

Hence $\lim_{n\to\infty} Bz_{2n} = sz$. We claim that $Az = Sz$. If $Az \ne Sz$, then by virtue of (iii), for sufficiently large values of n, we get

$$d(Az, Bz_{2n}) < \max\{k_1 d(Sz, Tz_{2n}), k_2[d(Az, Sz) + d(Bz_{2n} Tz_{2n})]/2,$$

$$[d(Sz, Bz_{2n}) + d(Az, Tz_{2n})]/2\}.$$

On letting $n \to \infty$, this yields $d(Az, Sz) < [d(Az, Sz)]/2 < d(Az, Sz)$, a contradiction. Hence $Az = Sz$. Also, since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. If $Bw \neq Tw$, using (iii) we get

$$d(Az, Bw) < \max\{k_1 d(Sz, Tw), k_2[d(Az, Sz) + d(Bw, Tw)]/2, [d(Sz, Bw)+d(Az, Tw)]/2\},$$

$$= [d(Bw, Tw)]/2 < d(Bw, Tw) = d(Bw, Az), \text{ a contradiction.}$$

Hence $Az = Bw$ and so, $Sz = Az = Tw = Bw$.

Since compatible maps commute at their coincidence points, we get $ASz = SAz$ and $BTw = TBw$. Moreover, $AAz = ASz = SAz = SSz$ and $BBw = BTw = TBw = TTw$. If $Az \neq AAz$, using (ii), we find

$$d(Az, AAz) = d(AAz, Bw) < \max\{d(SAz, Tw), d(AAz, SAz), d(Bw, Tw), [d(SAz, Bw) + d(AAz, Tw)]/2\},$$

$$= d(Bw, AAz) < d(AAz, Bw), \text{ a contradiction.}$$

So that $Az = AAz = SAz$ and so Az is a common fixed point of A and S. Similarly, $Bw\,(= Az)$ is a common fixed point of B and T. Uniqueness of the common fixed point follows from (ii). The proof is similar when T is assumed continuous in place of S. Moreover, since $AX \subset TX$ and $BX \subset SX$, the proof follows on similar lines when A or B is assumed to be continuous. This establishes the theorem.

We now give an example to illustrate the above theorem.

EXAMPLE2. Let $X = [2, 20]$ and d be the Euclidean metric on X. Define A, B, S and $T : X \to X$ as follows:

$Ax = 2$ for each x

$Bx = 2$ if $x < 4$ or ≥ 5, $\quad Bx = 3 + x$ if $4 \leq x < 5$;

$Sx = x$ if $x \leq 8$, $\quad Sx = 8 \quad$ if $x > 8$;

$Tx = 2$, if $x < 4$ or ≥ 5, $\quad Tx = 9 + x$ if $4 \leq x < 5$.

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It can be seen in this example that A, B, S and T

satisfy the condition (ii) when $\delta(\varepsilon) = 1$ if $\varepsilon \geq 6$ and $\delta(\varepsilon) = 6 - \varepsilon$ if $\varepsilon \geq 6$. Thus, $\delta(\varepsilon)$ is neither non decreasing nor lower semi continuous. It can also be verified that the mappings A, B, S and T do satisfy the contractive condition (iii) with $k_1 = 1$ and $k_2 = 1$. However, A, B, S, and T do not satisfy the ϕ-contractive condition (3) since the required function $\phi(t)$ can not be defined at $t = 6$. Hence we see that the present example does not satisfy the conditions of any previously known common fixed point theorem for contractive type mappings, since neither the mappings satisfy a ϕ-contractive condition nor δ is lower semi continuous or non decreasing.

REMARKS. As various assumptions either on ϕ or on δ have been considered to ensure the existence of common fixed points under contractive conditions, so our Theorem 1 improves the results of Boyd and Wong [1]. Carbone *et. al.* [2], Matkowski, [8], Pant [10, 11, 12], Pant and Pant [14], Park and Rhoades [17], Singh and Kashahara [20], Jungck [5], Jungck *et al.* [6]. Jachymski [4], Maiti and Pal [7], and Park and Bae [16] for the case when k_1=1 and k_2= 1. All such results are obtained as special case of Theorem 1 when k_1=1 and k_2 = 1. Also our theorem, thus, generalizes all other similar results for fixed points and allows k_1 to take values other than 1 by taking a Lipschitz type contractive condition.

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NON-EQUILIBRIUM THERMODYNAMIC MODEL OF ECOSYSTEM: CRITERIA OF ECOLOGICAL STABILITY

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ABSTRACT

The object of this paper is to develop a non-equilibrium thermodynamic model of a population ecosystem and to study the thermodynamic criteria of stability of the ecosystem.

Mathematics Subject Classification (2000) : 92A17

Key words: Ecosystems, Non-equilibrium thermodynamic model, Lyapunov function, entropy function, ecological stability.

INTRODUCTION

An ecosystem consisting of an enormous number of interacting elements in relation to its environment is a complex system [1, 2, 3]. The statistical mechanical model of ecosystems initiated by Kerner [4, 5] based on Gibbsian ensemble theory, although very powerful, has however some limitations. It is limited to conservative system, but most of the ecological model equations are of non-conservative type. In recent times great efforts are going on in developing general non-equilibrium thermodynamic models of complex ecosystems [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. The thermodynamics has the ability to provide a unified description of macroscopic systems which are independent of the details of ongoing processes in terms of state functionals depending on limited number of observables.

In an earlier paper [7] we have developed a non-equilibrium thermodynamic model of Lokta-Volterra ecosystems. The object of the present paper is to supplement the earlier one. In this objective our first task is to derive the expression for entropy-production of a general model ecosystem, which is independent of Lokta-Volterra model equations or any other model equations of biological and chemical reaction systems. The second is to explore the extent to which the thermodynamic model can provide useful characterization of the dynamical complexity of stability of some model ecosystems.

ECOSYSTEM : THERMODYNAMIC MODEL

In applying thermodynamics to the theory of ecosystem we assume that biological phenomena, although not completely determined by the laws of macroscopic physics and chemistry, are not in contradiction to them [14]. For a consistent thermodynamic theory of ecosystems, we make the following assumptions:

1. We assume the ecosystem as a regional community of biological species (excluding biotic components) embedded in an environment.

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2. Let N_i, $(i = 1, 2, \ldots, n)$ be the population of the i-th species and N_i is assumed to he governed by the system of Kolmogorov's type of equations:

$$\frac{dN_i}{dt} = N_i f_i(N_1, N_2, \ldots, N_n), \qquad (i = 1,2,\ldots n) \tag{2.1}$$

where the functions F_i's are assumed to be continuous function of $(N_1, N_2, \ldots, N_n)$ having partial derivatives in some open set $\Omega = [N_i \geq 0 : i = 1,2,\ldots,n]$.

3. We assume that the energy of the ecosystem is stored exclusively within the organisms so that the transfer or flow of energy is governed exclusively by the system of dynamical equations characterizing the model ecosystem.

4. The model equations are taken as the basic equations for the non-equilibrium thermodynamic modelling of the ecosystem. This is analogous to Kerner's statistical mechanical modelling of ecosystem on the basis of Lotka-Volterra equations.

5. The state of ecosystem is assumed to be defined by the set of population sizes $N = [N_1, N_2, \ldots, N_n]$ and they are assumed to be extensive variables for the thermodynamic description of the system.

Let $N^* = [N_1^*, N_2^*, \ldots, N_n^*]$ be the stationary values of $N = [N_1, N_2, \ldots, N_n]$. Let S be the entropy of the system at any time t and is assumed to be a function of the state (or extensive) variables $[N_1, N_2, \ldots, N_n]$, that is, $S = S(N_1, N_2, \ldots N_n)$. Let ΔS be the change of entropy of the system in transition from the stationary state $N^* = [N_1^*, N_2^*, \ldots, N_n^*]$ to the non-stationary state $N = [N_1, N_2, \ldots, N_n]$, that is,

$$\Delta S = S - S_{st} \tag{2.2}$$

Let us now develop a non-linear thermodynamic model of the ecosystem described by the dynamical equation (2.1). Expanding $f_i(N_1, N_2, \ldots, N_n)$ about the stationary state $[N_1^*, N_2^*, \ldots N_n^*]$ we can reduce the system of equation (2.1) to the form

$$\frac{1}{N_i}\frac{dN_i}{dt} = f_i(N_1^*, N_2^*, \ldots, N_n^*) + \sum_{j=1}^{n}\left[\frac{\partial f_i}{\partial N_j}\right]_{st}(N_j - N_j^*)$$

$$+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{\partial^2 f_i}{\partial N_j \partial N_k}\right]_{st}(N_j - N_j^*)(N_k - N_k^*) + \ldots \tag{2.3}$$

Neglecting higher-order terms of $\delta N_j = N_j - N_j^*$, we have

$$\frac{1}{N_i}\frac{dN_i}{dt} = \sum_{j=1}^{n}\left[\frac{\partial f_i}{\partial N_j}\right]_{st}\left(N_j - N_j^*\right) \tag{2.4}$$

Let us now define the thermodynamic fluxes J_i the specific growth rate

$$J_i = \frac{1}{N_i}\frac{dN_i}{dt} = \frac{d}{dt}(\ln N_i), (i = 1,2,....,n) \tag{2.5}$$

and the corresponding thermodynamic forces X_i as

$$X_i = N_i - N_1^*, (i = 1,2,..,n) \tag{2.6}$$

which are the deviation of the macroscopic or state-variables N_i from their stationary value N_i. With these definitions of thermodynamic fluxes, j_i and forces X_i the system of equations (2.4) reduces to the form of lincar phenomenological relations of irreversible thermodynamics [10] :

$$j_i = \sum_{j=1}^{n} L_{ij}X_j, (i = 1,2,....,n) \tag{2.7}$$

where

$$L_{ij} = \left[\frac{\partial f_i}{\partial N_j}\right]_{st} \quad (i,j = 1,2,...,n) \tag{2.8}$$

are the phenomenological coefficients. We now find the explict form of the entropy-production ΔS. According to Onsagar the entropy-production is given by [10]

$$\frac{dS}{dt} = \sum_{i=1}^{n} X_i J_i = \sum_{i=1}^{n}(N_i - N_1^*)\frac{d}{dt}(lnN_i) = \sum_{i=1}^{n}\frac{N_i - N_i^*}{N_i}\frac{dN_i}{dt} \tag{2.9}$$

Integrating (2.9) from the stationary state $[N_1^*, N_2^*, ..., N_N^*]$ to the non-stationary state $[N_{1,} N_2 ...., N_n]$, we have

$$\Delta S = \sum_{i=1}^{n}\left[\left(N_i - N_1^*\right) + N_1^* ln\frac{N_1^*}{N_1}\right] \tag{2.10}$$

which is the expression of the change of entropy (except a multiplicative constant) in transition from the stationary state $[N_1^*, N_2^*, ..., N_n^*]$ to the non-stationary state $[N_1, N_{2,} ...., N_n]$. The expression (2.10) plays a crucial role in the study of stability of the system.

ENTROPY AND LYAPUNOV FUNCTION : THERMODYNAMIC CRITERIA OF STABILITY

A class of Lyapunow function for the dynamical system (2.1) is given by [17]

$$L = \sum_{i=1}^{n} N_i^{*} \varnothing\left(\frac{N_i}{N_1^{*}}\right) \tag{3.1}$$

where $\varnothing$ is a convex function of N_i having isolated minimum at $N_i = N_1^{*}$ and increasing monotonically. Now by Lyapunov theorem [17] the stationary state $[N_1^{*}, N_2^{*}, \ldots., N_n^{*}]$ is stable if $dL/dt \geq 0$ taken along the trajectory of (2.1) and unstable if $dL/dt > 0$ even for one trajectory of (2.1). If we consider the function $\varnothing$ in (3.1) by

$$\varnothing(\xi_i) - \text{In}\,\xi_i - 1, \text{ where } \xi_i = \frac{N_i}{N_1^{*}} \tag{3.2}$$

Then the corresponding Lyapunov function will be given by

$$L = \sum_{i=1}^{n}\left[(N_i - N_i^{*}) In \frac{N_i^{*}}{N_i}\right] \tag{3.3}$$

which is the same expression of entropy change (2.10) derived from the non-equilibrium thermodynamic modelling of the dynamical system (2.1). The entropy change ΔS given by (2.10) which is positive and vanishes identically for $N_i = N_i^{*}, (i = 1,2,\ldots,n)$ is this a Lyapunov function defined over the state space $[N_1, N_2, \ldots, N_n]$ of the system. For the system of model equations (2.1) and the entropy change ΔS as the Lyapunov we have the following theorem:

Theorem : The model system desribed by the system of equations (2.1) is globally stable in the positive orthant of the state space $[N_1, N_2, \ldots, N_n]$ if the time derivative along the trajectory

$$\frac{d}{dt}(\Delta S) = \sum_{i=1}^{n}(N_i - N_i^{*}) f_i(N_1, N_2, \ldots, N_n) \leq 0 \tag{3.4}$$

in the positive orthant and does not vanish identically along any sholution of the system (2.1) other than the stationary state $[N_1^{*}, N_2^{*}, \ldots, N_n^{*}]$.

The proof follows directly from LaSalle's extension of the direct. Lyapunov theorem. The equality in (3.4) corresponds to the marginal stable state having indeterminate stability characteristing stable from the unstable regions [17].

Let us now derive the thermodynamic criteria of stability from the above general criteria (3.4). Applying Taylor's series expansion of $f_i(N_1, N_2, \dots, N_n)$ and using (2.8) and (2.9), we have

$$\frac{d}{dt}(\Delta S) = \sum_{j=1}^{n}\sum_{k=1}^{n}(N_j - N_j^*)\left[\frac{\partial f_j}{\partial f_k}\right]_{st}(N_k - N_k^*) = \sum_{j=1}^{n}\sum_{k=1}^{n} X_j L_{jk} X_k \tag{3.5}$$

The criteria of stability of the stationary state $[N_1^*, N_2^*, \dots, N_n^*]$ then becomes the negative-definiteness of the quadratic form (3.5) or of the matrix $[L_{jk}]$ of the phenomenological coefficients. In the next section we shall test this criterion of stability for some model ecosystems.

Let usnow consider the stability of Lotka-Volterra ecosystem in the light of the above criteria of stability. We consider an ecosystem governed by the generalized Lotka-Volterra system of equation [7]

$$\frac{dN_i}{dt} = N_i\left[a_i - \sum_{j=1}^{n} b_{ij}N_j\right], \quad (i = 1,2,\dots,n) \tag{3.6}$$

where $a_i, (i = 1,2,\dots,n)$ may have any sign and there is no restriction on the coefficients $b_{ij}, (i, j = 1,2,\dots,n)$. The non-trivial stationary state $[N_1^*, N_2^*, \dots, N_n^*]$ of the system (3.6) is given by

$$\sum_{j=1}^{n} b_{ij}N_j^* = a_i, \quad (i = 1,2,\dots,n) \tag{3.7}$$

The necessary and sufficient condition for the uniqu solution of (3.7) with all N_j^* positive is that the number of species n must be even and that all $a_i's$ be not of the same sign [18]. With the help of (3.7) the system of equation (3.6) can be written as

$$\frac{1}{N_i}\frac{dN_i}{dt} = \frac{d}{dt}(InN_i) = \sum_{j=1}^{n} b_{ij}(N_j^* - N_j) \tag{3.8}$$

The right hand side of (3.8) is already a linear function of $[N_1, N_2, \dots, n]$. Comparing with (2.7), we see that

$$L_{ij} = b_{ij}, \quad (i, j = 1,2,\dots,n) \tag{3.9}$$

so that the criteria of stability of the stationary state $[N_1^*, N_2^*, \dots, N_n^*]$ for the generalized

Lotka-Voltera system (3.6) is the positive definiteness of the matrix $\left[b_{ij}\right]$ [18]. The Lotka-Volterra system (3.6) written in the form (2.1) is a special form of the system, the functions $f_i(N_1, N_2, \ldots, N_n)$ in the right hand side of (2.1) are simply linear function of the state variable. But all ecological systems are not of Lotka-Volterra type and the functions $f_i(N_1, N_2, \ldots, N_n)$ are in general non-linear. In the next section we shall consider these two types of systems namely Lotka-Volterra type and non-Lotka-Volterra type to test the thermodynamic criteria of stability.

APPLICATIONS : STABILITY OF SOME MODEL ECOSYSTEMS

In this section we shall study the problems of stability of some model ecosystems on the basis of the theories developed in the previous section.

A. Competitive system of two species:

let us first consider a two species Lotka-Voltka competitive model ecosystem described by the system of equations

$$\frac{dN_1}{dt} = N_1 F_1(N_1, N_2) \qquad \frac{dN_2}{dt} = N_2 F_2(N_1, N_2) \tag{4.1}$$

We linearize the system of equations (4.1) about he stationary state (N_1^*, N_2^*). Proceeding as before, we have the system of phenomenological equations

$$J_1 = L_{11}X_1 + L_{12}X_2, \qquad J_2 = L_{21}X_1 + L_{22}X_2 \tag{4.2}$$

where

$$L_{ij} = \left[\frac{\partial F_i}{\partial N_j}\right]_{st} \qquad i, j = 1,2 \tag{4.3}$$

The criteria of stability of the stationary state (N_1^*, N_2^*) is given by, $L_{11} < 0, L_{11}L_{22} - L_{12}L_{21} > 0$. This last condition can be written in terms of partial derivatives of $F_1(N_1, N_2)$ and $F_2(N_1, N_2)$ with respect to N_1 and N_2 evaluated at (N_1^*, N_2^*) as follows :

$$\left[\frac{\partial F_1}{\partial N_1}\right]_{st}\left[\frac{\partial F_2}{\partial N_2}\right]_{st} > \left[\frac{\partial F_1}{\partial N_2}\right]_{st}\left[\frac{\partial F_2}{\partial N_1}\right]_{st} \tag{4.4}$$

This is the criterion of competetive exclusion introduced first by Gilpin and Justice [19] without any proof. Let us now consider in particular a competetive ecosystem of the two species governed by the system of equations

$$\frac{dN_1}{dt} = N_1(a - bN_1 - cN_2) \tag{4.5a}$$

$$\frac{dN_2}{dt} = N_2(e - fN_1 - gN_2) \tag{4.5b}$$

where a,b,c,e,f,g are positive constants. The stationary states (N_1^*, N_2^*) are given by,

(i) $(N_1^*, N_2^*) = (0,0)$, (both the species die out).

(ii) $(N_1^*, N_2^*) = (0,e/g)$, (first species die out).

(iii) $(N_1^*, N_2^*) = (a/b,0)$, (second species die out).

(iv) $(N_1^*, N_2^*) = \left(\frac{ag-ec}{gb-cf}, \frac{af-be}{fc-bg}\right)$, (both species exit).

defining the thermodynamic fluxes J_i and forces X_i as in (2.5) and (2.6), the system of equations (4.5) can be reduced to the form of linear phenomenological relations :

$$J_1 = L_{11}X_1 + L_{12}X_2, \qquad J_2 = L_{21}X_1 + L_{22}X_2$$

where

$$L_{11} = \left[\frac{\partial F_1(N_1, N_2)}{\partial N_1}\right]_{st} = -b \tag{4.6a}$$

$$L_{12} = \left[\frac{\partial F_1(N_1, N_2)}{\partial N_2}\right]_{st} = -c \tag{4.6b}$$

$$L_{21} = \left[\frac{\partial F_2(N_1, N_2)}{\partial N_1}\right]_{st} = -f \tag{4.6c}$$

$$L_{22} = \left[\frac{\partial F_2(N_1, N_2)}{\partial N_2}\right]_{st} = -g \tag{4.6d}$$

In this case L_{ij}'s are independent of different values of (N_1^*, N_2^*). Here $L_{11} = -b < 0$ as 'b' is positive. Now, $L_{11}L_{22} - L_{12}L_{21} > 0$, that is of $bg > cf$.

B. A Prey-Predator System :

As a second illustative example let us consider a non-Lotka-Volterra of model prey-predator system described by the system of equations [20,21]

$$\frac{1}{N_1}\frac{dN_1}{dt} = N_1(1-N_1) - N_2 = F_1(N_1, N_2) \qquad (4.7a)$$

$$\frac{1}{N_2}\frac{dN_2}{dt} = k(N_1 - \mu) = F_1(N_1, N_2) \qquad (4.7b)$$

where N_1 and N_2 are the populations of the prey and predator species respectively. The stationary states (N_1^*, N_2^*) are given by

(i) $(N_1^*, N_2^*) = (0,0)$, (both prey and predator die out).

(ii) $(N_1^*, N_2^*) = (1,0)$, (only predator species die out).

(iii) $(N_1^*, N_2^*) = (\mu, \mu - \mu^2)$, (both prey and predator species coexist).

Defining the thermodynamic fluxes J_i and fores X_i as in (2.5) and (2.6), the system of equations (4.7) can be reduced to the form of linear phenomenological relations

$$J_1 = L_{11}X_1 + L_{12}X_2, \qquad J_2 = L_{21}X_1 + L_{22}X_2$$

where

$$L_{11} = \left[\frac{\partial F_1(N_1, N_2)}{\partial N_1}\right]_{st} = 1 - 2N_1^* \qquad (4.8a)$$

$$L_{12} = \left[\frac{\partial F_1(N_1, N_2)}{\partial N_2}\right]_{st} = -c \qquad (4.8b)$$

$$L_{21} = \left[\frac{\partial F_2(N_1, N_2)}{\partial N_1}\right]_{st} = -f \qquad (4.8c)$$

$$L_{22} = \left[\frac{\partial F_2(N_1,N_2)}{\partial N_2}\right]_{st} = -g \tag{4.8d}$$

Let us now test the criteria of stability for different steady- states. The steady-state (0,0) is trivial and unstable. We consider the second steady-state (1,0), for this the phenomenological coefficients L_{ij} are given by $L_{11} = -1, L_{12} = -1, L_{21} = k, L_{22} = 0$. Here $L_{11} < 0, L_{11L}L_{22} - L_{12}L_{21} = k$. So the steady-state (1,0) is stable $k > 0$. Let us now test the steady-state $(\mu, \mu - \mu^2)$. For this interior steady-state L_{ij} is given by

$$L_{11} = 1 - 2\mu, L_{12} = -1, L_{21} = k, L_{22} = 0$$

So the steady-state $(\mu, \mu - \mu^2)$ is stable if $L_{11} = 1 - 2\mu < 0$ *or* $\mu < 0.5$ and $L_{11}L_{22} - L_{12}L_{21} = k > 0$. Thus for $k > 0$ the steady-state $(\mu,\ \mu - \mu^2)$ is a stable for $\mu > 0.5$ and unstable if $\mu > 0.5$. For $\mu = 0.5, k = 1$ the state is marginally stable. In fact, the parametric value $\mu = 0.5$ is a Hopf-bifurcation point [21,22].

CONCLUSION

The concept of stability is very fundamental in the study of biological communities and there are a lot of theories on stability for ecological communities [18, 23, 24]. A community is considered stable when the component populations do not undergo sharp fluctuation and this definition is closer to the thermodynamic (or more correctly to the statistical physics) notion of system stability. Evidently general thermodynamic concepts (for instance the second law of thermodynamics for closed syste and its extension for open systems) should be applicable to biological and in particular, to ecological system. In fact, the different thermodynamic quantities like entropy, thermodynamic potential etc. are Lyapubnov function and the stability analysis based on these thermodynamic quantities are special cases of general dynamical analysis of stability based on Lyapunov function [25]. The present paper is an attempt in this direction. On the basis of the thermodynamic model of a general ecological system we have derived the thermodynamic criteria of ecological stability and tested these criteria for some model ecosystems.

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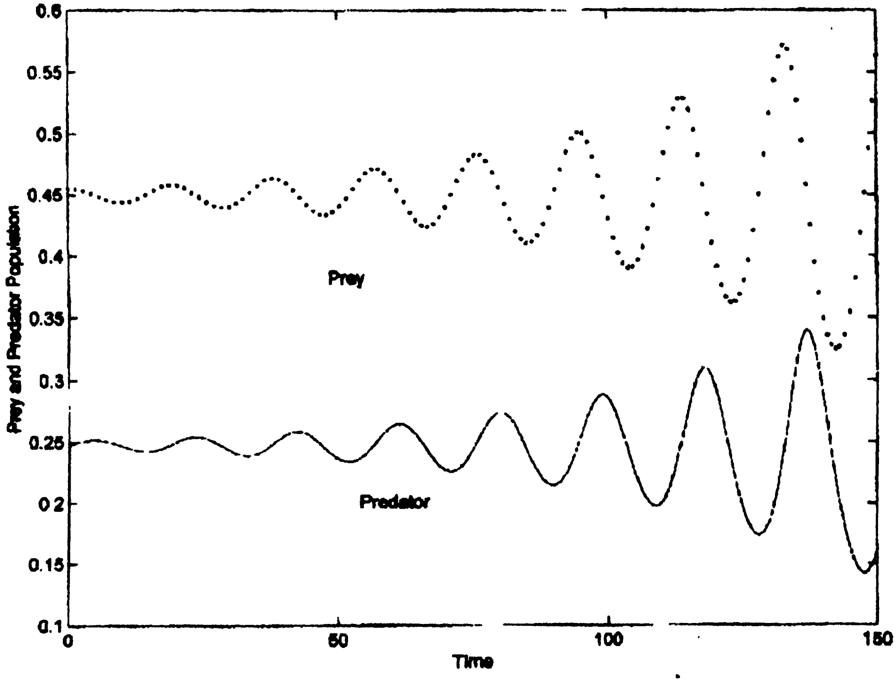
Prey and Predator Population
Prey
Predator
Time
0.6
0.55
0.5
0.45
0.4
0.35
0.3
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0.15
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50
100
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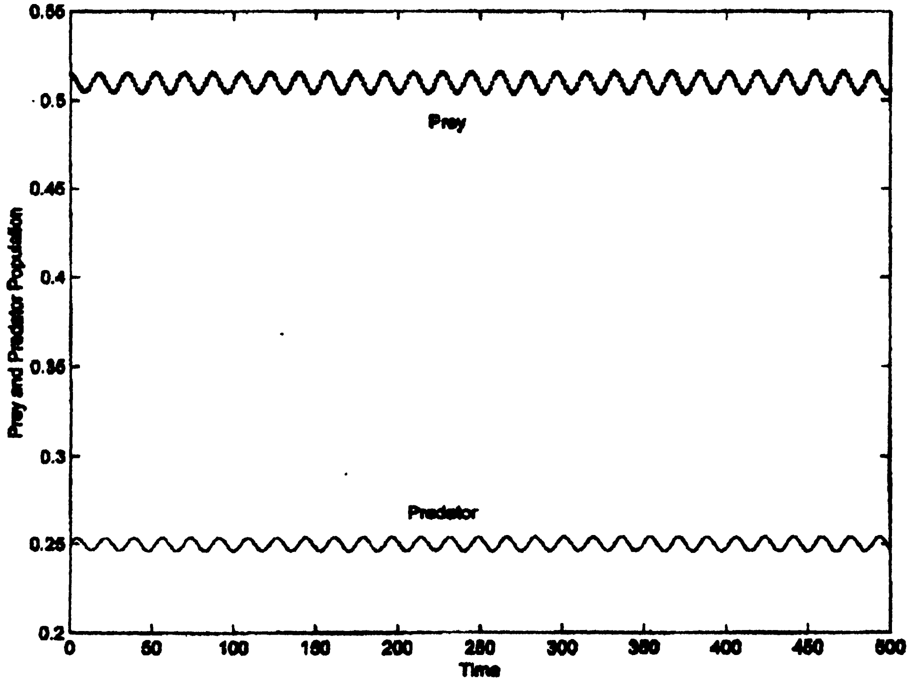
Prey and Predator Population
Prey
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0.55
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PERIODATE OXIDATION OF O-CHLOROANILINE – A KINETIC MECHANISTIC STUDY

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ABSTRACT

The kinetic-mechanistic studies related to the periodate oxidation of o-chloroaniline in acetone-water medium have been reported. The studies were made by employing the spectrophotometric technique. The order w.r.t. both oxidant and substrate has been found to be one in each. The rate increases on increasing the ionic strength and decreases with decrease in dielectric constant. The rate-pH profile has been discussed. The effect of free radical scavengers rejects the possibility of a free radical mechanism. The thermodynamic parameters have been evaluated and discussed. The main product of oxidation characterized by melting point and UV-VIS, I.R. and N.M.R. spectrum was chloro-1,4 benzoquinone. A suitable mechanism has been proposed and the rate law derived. 1 mol of the substrate was found to react with 2 moles of the oxidant.

Key word & phrases: Periodate oxidation ; o-chloroaniline; kinetics and mechanism.

INTRODUCTION

The kinetic studies made on the non-Malapradian oxidation of aromatic amines by periodate ion are rather few [4,7,11,12,14]. In continuation to earlier studies [8,9], the resutls of the periodate oxidation of o-chloroaniline (OCA) are being reported and discussed.

MATERIALS AND METHODS

OCA and sodium metaperiodate of E.Merck A.R. grade were used after Zn-dust distillation/recrystallization. Doubly distilled water and other chemicals of CDH A.R. grade were used. Thiel, Schultz and Koch buffer [2] was used for maintaining the pH of the reaction mixtures.

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The progress of the reaction was followed by recording the absorbances of the light orange colored reaction mixture on Schimadzu double beam spectrophotometer, UV-150-02 at the wavelength 460 nm i.e. the λ_{max} of the reaction mixture, keeping the pH at 5.0 (except for the kinetic runs in which the effect of pH was being studied) during the period in which the λ_{max} did not change. Initial rates in terms of $(dA/dt)_i$ were evaluated by plane mirror method while the pseudo first order rate constant k_1, was calculated by using Guggenheim's method. The second order rate constants k_2 were obtained by dividing the k_1 by [S] wehre S is the reactant taken in excess. The pH of solutions was measured by using Systronics digital pH meter model-335. A high precision thermostatically controlled water bath with accuracy ± 0.1°C was employed for maintaining the constant temperature during the kinetic runs. The stoichiometry of the reaction was determined iodometrically.

RESULTS AND DISCUSSION

Reaction mixture containing 0.001 M [OCA] + 0.02 M [$NaIO_4$] + 2.5 % (v/v) acetone, was prepared and kept for 24 hours. It developed violet colour changing into orange followed by precipitation. It was filtered after 24 hours, filtrate extracted with petroleum ether and evaporated at room temperature. On separation of a droplet (due to water), a solid residue was left on the petri dish. This solid residue was dissolved in petroleum ether (40-60°C) and a little amount of silica gel was added to it and mixed with a clean spatula so as to absorb the complete residue on silica gel. This silica gel was then applied on the top of a silica gel-H packed, 31 cm long column with 0.6 cm diameter. Gradient elution was employed using petroleum ether and chloroform in decreasing ratio. Four components were separated out of which a light orange colored major component was collected, checked for purity by using TLC and crystallized in ethyl alcohol. Its melting point was found to be 59° C. This compound was identified as a quinone [5]. The UV-VIS spectrum of this compound in $CHCl_3$, showed λ_{max} for this compound at

270 nm, 370 nm and 520 nm, which suggested the presence of quinonoid structure.

Further, I.R. spectrum (in KBr) showed bands at 1632 cm^{-1}(s) (due to 1,4-benzoquinone pattern); 3255 cm^{-1}(s) (due to overtone of C=O strech); 2713 cm^{-1}(w) (due to isolated C-H streching); 1514 cm^{-1}(s), 1469 cm^{-1}(s) and 1354 cm^{-1}(s) (due to C=C ring strech); 1200 cm^{-1}(s) and 1098 cm^{-1} (due to in plane C-H bending); 825 cm^{-1}(m) (due to out to plane C-H bending in case of two adjacent H-atoms); 761 cm^{-1}(s) (a characteristic strong band due to C-Cl streching); and 612 cm^{-1}(m) and 518 cm^{-1}(m) both due to C-H bending mode). This product was finally charactrized as chloro-1,4-benzoquinone on the basis of its N.M.R. spectrum in $CDCl_3$ which showed the peaks at $\delta = 7.260$ (s, 1H) and at $\delta = 6.991$ (d, 2H) which are due to three protons of the ring [3,10,13].

Stoichiometry of the reaction was found to be I mol OCA : 2 mole periodate. The data presented in the table-1 were subjected to calculations by using the Vant hoff's differential equation and it was established that the reaction is second order,being first order in each reactant. The second order kinetics was also proved by the fact that the rate was linearly related to the concentration of the reactant varied in each case. Under pseudo first order conditions, the values of $(dA/dt)^{-1}$ vs $[S]^{-1}$ (Table-2) were also used to get a linear plot with almost negligible intercept, indicating that the intermediate formed in the slow step, should be unstable and it should have got consumed in a subsequent fast step.

Table-1

$\lambda_{max} = 460$nm; pH = 5.0; *Acetone = 5.0%(v / v); ≠ Acetone = 20%(v / v); Temp. = 30 ± 0.1^{0}C

[OCA] x 10^3 M	50.0≠	50.0≠	50.0≠	50.0≠	50.0≠	50.0≠	3.0*	4.0*	5.0*	6.0*	7.0*	8.0*
$[NaIO_4]$ x 10^3 M	5.0	6.0	7.0	8.0	9.0	10.0	30.0	30.0	30.0	30.0	30.0	30.0
$(dA/dt)_i$ x 10^3 (min^{-1})	0.70	0.85	0.95	1.00	1.25	1.375	3.3	3.8	4.8	6.0	7.0	8.0

Table-2

λ_{max} = 460nm; pH = 5.0; *Acetone = 25.0%(v / v); ≠ Acetone = 50%(v / v); Temp. = 30 ± 0.1°C

OCA x 10^2 M	4.5*	5.2*	5.4*	5.6*	5.8*	6.0*	0.3≠	0.3≠	0.3≠	0.3≠	0.3≠	0.3≠
[$NaIO_4$] x 10^2M	0.5	0.5	0.5	0.5	0.5	0.5	3.0	3.2	3.4	3.6	3.8	4.0.
$(dA/dt)_i$ x 104 (min-1)	3.4	3.5	3.6	3.8	3.9	4.0	3.3	3.5	3.8	4.0	4.25	4.5

Kinetic studies were also made in the pH range 3.5 range 3.5 to 8.0 (Table-3) indicated that the rate was maximum at pH 5.0 The first part of the profile i.e. the increase in the rate from pH 3.5 to 5.0 may be to the decrease in the protonation of OCA from pH 3.5 to 5.0, which makes greater concentrations of OCA available for the reaction. This assumption is in lin with the fact that unprotonated OCA is the reactive species as shown by us in the mechanism proposed. Further, the concentration of the periodate monoanion is maximum around pH 5.0 and decreases beyond this pH value as worked out by earlier workers [4,6,11,15,16]. This may be the reason for the decrease in the rate of reaction beyond pH 5.0 This behavior also supports our assumption in the propsed mechanism (CHART) that out of the various species of the periodate, the species taking part in the reaction in present case is the periodate monoanion i.e. [IO_4].

Table-3. Effect of pH on the reaction rate

[OCA] = 3.0 x 10^{-3} M, [$NaIO_4$] = 3.0x10^{-2} M, λ_{max} = 460 nm Acetone = 5.0 % (v/v), Temp. 35.0 ± 0.1 1°C.

pH	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
$(dA/dt)_i$ x 104 (min^{-1})	20.0	25.0	30.0	40.0	30.0	25.0	20.0	15.0	10.0	5.0

The data in table - 4, suggests a linear relation between log $(dA/dt)_i$ and 1/D (where D is the dielectric constant of the medium) with negative slope indicating an ion-dipole interaction in this reaction [1]. This point is well supported by the high negative value of the entropy of activation and our observation that the free radical scavengers like acry-amide and allyl alcohol exerted no effect on the reaction rate. A primary linear type plot between log $(dA/dt)_i$ vs ionic strength (μ) was obtained by us by using the data given in the table-5, which also supports the ion-dipole interaction assumed by us in the mechanism proposed.

Table - 4. Effect of dielectric constant of the medium on the reaction rate

[OCA] = 3.0 x 10^{-3} M, [$NaIO_4$] = 3.0 x 10^{-2} M, λ_{max} = 460 nm; pH = 5.0, Temp. = 30 ± 0.1°C				
Dielectric constant	72.4	71.0	69.7	68.4
(dA/dt)i x 103 (min^{-1})	3.75	2.4	1.6	1.0

Table - 5. Effect of dielectric constant of the medium on the reaction rate

[OCA] = 3.0 x 10^{-3} M, [Na I O4] = 3.0 x 10^{-2} M, λ_{max} = 460 nm; pH = 5.0; Acetone = 5.0% (v/v); Temp. = 35.0 ± 0.1°C.				
μ x 10^2	4.0	5.0	6.0	7.0
$(dA/dt)_i$ x 10^3 (min^{-1})	3.5	4.0	4.5	5.25

The values of different thermodynamic parameters evaluated from the linear Arrhenius plot obtained by using the results of the kinetic studies made at four temperatures ranging from 30 ± 0.1° C to 45 + 0.1° C and taking [OCA] = - 0.003 M, [NaI O_4] = 0.03 M and acetone = 10.0% (v/v) are, Ea = 10.94 k cal. mol^{-1}; A = 1.692 x 10^6 lit. mol^{-1}. sec-1 ; $\Delta S^{\neq}$ =- 32.11 e.u. ; ΔF^{*} = 20.295 k.cal.mol^{-1} and $\Delta H^{\neq}$ = 10.325 k.cal. mol^{-1}. Where, Ea is the energy of activation, A is the Arrhenius frequency factor, ΔS^{*} is the entropy of activation, ΔF^{*} is the free energy of activation and ΔH^{*} is the enthalpy of activation.

A large negative value of entropy of activation is mainly observed in polar solvents and also suggests the formation of a charged and rigid transition state which is expected to be strongly solvated in the polar solvent employed and is in accordance with the first and third steps in the mechanism proposed. The decrease in rate with the decrease in dielectric constant (Table-4) also supports this assumption. A low value of energy of activation is characteristic of a bimolecular reaction in the solution.

On the bais of kinetic studies and the main reaction product characterized, the mechanism presented in the chart can be proposed which fits the rate law given below.

$$(dA/dt) = k_2\,[OCA][IO_4]$$

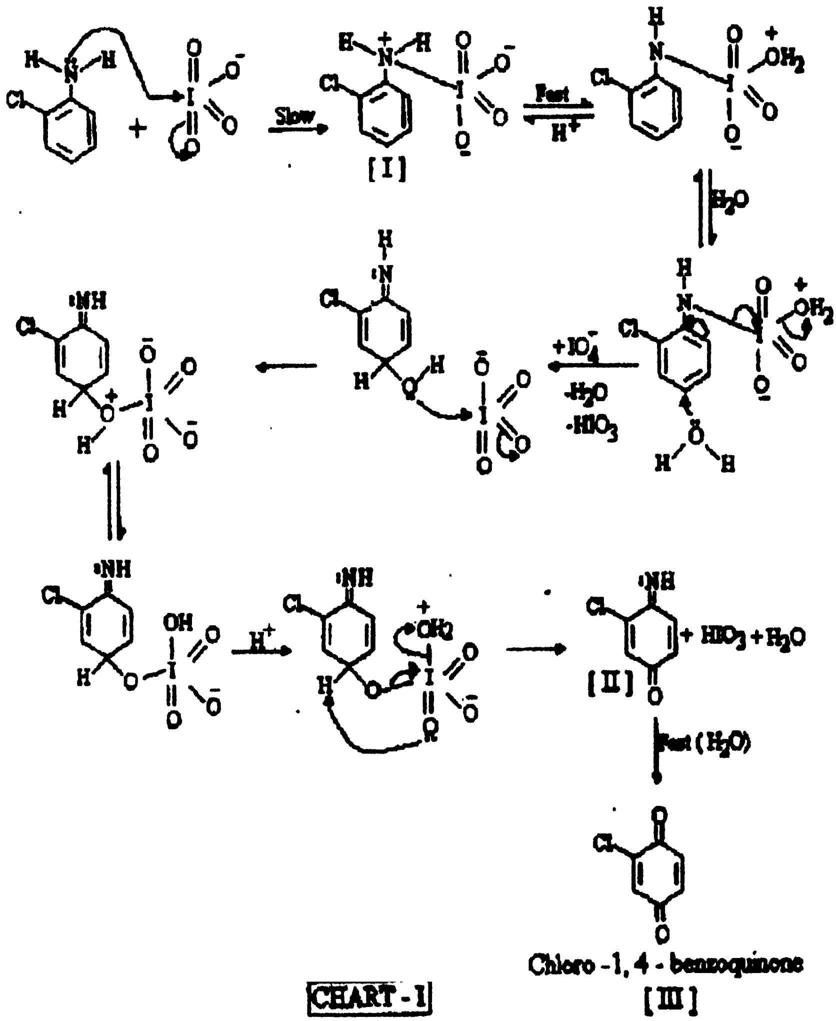

CHART - I

The formation of a charged intermediate (I) which reacts with another periodate molecule to form quinoneimine (II) and subsequent fast hydrolysis of II to give chloro-1,4-benzoquinone are the main features of the proposed mechanism is in good agreement with the reaction product characterized by us and the different kinetic results reported and discussed above viz. the order of reaction in each reactant, stoichiometry observed and the studies related to the effect of solvent, ionic, strength, pH, and free radical scavengers on the reaction rate as well as the thermodynamic parameters evaluated.

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QUASI-CONTRACTIONS AND APPROXIMATE FIXED POINTS

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ABSTRACT

In this note, we discuss approximate fixed points of quasi-contractions on metric spaces.

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Key words and phrases: Fixed point, approximate fixed point, contraction, quasi-contraction.

INTRODUCTION

Let (X, d) be a metric space and $T: X \to X$. Let ε be a positive number. Then a $z \in X$ is an ε-fixed point of if $d(Tz, z) \leq \varepsilon$. The map T is said to have approximate fixed point property if, for each $\varepsilon > 0$, the map possesses at least one ε-fixed point (see, for instance, [10]). The theory of approximate fixed points finds applications in mathematical economics, noncooperative game theory, dynamic programming, nonlinear analysis, variational calculus, theory of integro-differential equations and several other areas of applicable analysis (see, for instance, [1], [3], [4], [8], [10] and several references thereof).

The map T is a Banach contraction if

$d(Tx,Ty) \leq qd(x,y)$ for some $0 \leq q < 1$ and all x, y in X. (1)

The map T is called a quasi-contraction (cf. C'iric' [2]) if

$$d(Tx, Ty) \leq q. \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{2}$$

for some $0 \leq q < 1$ and all x, y in X.

We remark that T satisfying

$$d(Tx, Ty) \leq q.\max \{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\} \tag{3}$$

for some $0 \leq q < 1$ and all x, y in X need not satisfy (2). Notice that (1) $\Rightarrow$ (3) $\Rightarrow$ (2), and the reverse implication is not true. Indeed, among several generalizations of the Banach contraction, the quasi-contraction is considered the most general. For an excellent comparison of such maps, refer to Rhoades [9]. C'iric' [*op. cit.*] has shown that a quasi-contraction on a T-orbitally complete metric space has a unique fixed point. A metric space is called T-orbitally complete if every Cauchy sequence consisting of

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elements of the Picard orbit $O(x_0,\infty)$: $=\{x_0, Tx_0, T^2x_0, \ldots\}$ of T for any $x_0 \in X$ is convergent to a point in X. It is well known that a quasi-contraction need not be continuous, while a Banach contraction is uniformly continuous.

In all that follows q satisfying (1) or (2) will be called Banach contraction or quasi-contraction constant as the case may be. Recently Tijs et al. [10] have obtained the following approximate fixed point theorem for the Banach contraction.

Theorem 1. Let (X, d) be a metric space and T a contraction on X with the Banach contraction constant q. Then, for each $\varepsilon > 0$, $\text{FIX}_\varepsilon(T)$: $\{x \in X: d(Tx, x) \leq \varepsilon\} \neq \emptyset$ and the diameter of $\text{FIX}_\varepsilon(T)$ is not larger than $2\varepsilon/(1-q)$.

The purpose of this note is to extend the above result to quasi-contractions and discuss its implications.

APPROXIMATE FIXED POINT THEOREMS

The following is our main result.

Theorem 2. Let (X, d) be a metric space and T a quasi-contraction on X with quasi-contraction constant q. Then for each $\varepsilon > 0$,

(I) $\text{FIX}_\varepsilon(T) = \{x \in X: d(Tx, x) \leq \varepsilon\} \neq \emptyset$, and (II) the diameter of $\text{FIX}_\varepsilon(T)$ is not larger than $\left(\frac{2+q}{1-q}\right)\varepsilon$.

Proof. Pick $x_0 \in X$. Construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$. Then following C'iric' [2, p. 271], we have

$$d(T^n x_0, T^m x_0) \leq \left(\frac{q^n}{1-q}\right) d(x_0, Tx_0).$$

With $m = n + 1$ and n large, this implies that

$$d(x_n, x_{n+1}) \leq \left(\frac{q^n}{1-q}\right) d(x_0, x_1) < \varepsilon.$$

Therefore, for large n, x_n is an ε-fixed point of T. This proves (I).

Let w and z be any two ε-fixed points of T, then

$$\begin{aligned} d(w,z) &\leq d(w, Tw) + d(Tw, Tz) + d(z, Tz) \leq 2\varepsilon + d(Tw, Tz) \\ &\leq 2\varepsilon + q.\max\{d(w, z), d(w, Tw), d(z, Tz), d(w, Tz), d(z, Tw)\} \\ &\leq 2\varepsilon + q\{d(w, z) + \varepsilon\}. \end{aligned}$$

This yields (II).

Corollary 3. Let the quasi-contraction T in Theorem 2 possess a fixed point z. Then:
(I) z is the unique fixed point of T; and
(II) For each sequence $\{z_n\}$ with the property that for each $n \in N$ (naturals) the point z_n is an $1/n$-fixed point, we have $\lim_{n\to\infty} z_n = z$.

Proof. (I) is immediate from (2). Since z is a fixed point, it must belong to $\text{FIX}_\varepsilon(T)$ for each $\varepsilon > 0$. Then, by Theorem 2 (II),

$$d(z_n, z) \leq \text{diam } (\text{FIX}_{1/n}(T)) < (2+q)/\{n(1-q)\}.$$

So, $\lim_{n\to\infty} d(z_n, z) = 0$. This completes the proof.

We remark that neither the completeness nor the orbital completeness of the space is required in all the above results. The following example illustrates our result.

Example 4. Let $X = \{x: 0 \leq x < 1$ and x is a rational number$\}$ and $Tx = 1/2$, if $x = 1$ and 0, otherwise.

Then it is easily verified that T is a quasi-contraction with the fixed point 0, and $T \notin (1)$ or (3) (see also Rhoades [9, Theorem 1 (xxvi)]). Further, for any $0 < \varepsilon < 1/2$, $\text{FIX}_\varepsilon(T)$ consists of all points of $X \cap [0,1/2)$.

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